Harmonic maps from a two-torus into a complex projective space

Seiichi Udagawa

Department of Mathematics

Introduction

This paper is a exposition of recent result of Burstall's in [B] which says that any non-superminimal harmonic map from a two-torus into a complex projective space is covered by a *primitive* map of finite type into a certain generalized flag manifold. In [B-P], the outline of the proof of his result is appeared. First, it is observed that a primitive map ψ from a two-torus T^2 into a k-symmetric space G/K is of finite type if, for some (hence every) framing of ψ, $\alpha'_m(\partial/\partial z)$ is semisimple on a dense subset of T^2, where α'_m is defined as follows : For some (local) lifting $F : T^2 \longrightarrow G$ of $\psi : T^2 \longrightarrow G/K$, set $\alpha = F^{-1}dF$, which is the pull-back of the Maurer-Cartan form of G. Corresponding to the reductive decomposition $g = k + m$, where $m = T_o(G/K)$, set $\alpha = \alpha_m + \alpha_k$, and $\alpha_m = \alpha'_m + \alpha''_m$ is a decomposition into (1,0)-form and (0,1)-form, respectively.

In this paper, we give a proof of the fact that any non-superminimal harmonic map φ from a two-torus into a complex projective space may be lifted to a primitive map with semisimple α'_m into a certain generalized flag manifold, where the twistor space is chosen according to the isotropy order of φ.

1. Generalized flag manifold associated to a non-superminimal harmonic map into a complex projective space

Let $\varphi : S \longrightarrow \mathbf{C}P^n$ be a harmonic map, where $\mathbf{C}P^n = SU(n+1)/S(U(1) \times U(n))$. Let L_0 be the pull-back of universal bundle over $\mathbf{C}P^n$ by φ. L_0 is a subbundle of the trivial bundle $V(\mathbf{C}^{n+1}) = S \times \mathbf{C}^{n+1}$. We equip $V(\mathbf{C}^{n+1})$ with the standard Hermitian connected structure with Hermitian metric $< \, , \, >$ given by

$$< f, g > = \sum_{i=0}^{n} f_i \overline{g_i} \, , \quad \text{for} \;\; f = (f_0, f_1, \cdots, f_n), \;\; g = (g_0, g_1, \cdots, g_n)$$

For any subbundle F of $V(\mathbf{C}^{n+1})$, we denote by $F^{\perp}$ the Hermitian orthogonal complement of F in $V(\mathbf{C}^{n+1})$. Then, F and $F^{\perp}$ are both equiped with the induced Hermitian connected structures from $V(\mathbf{C}^{n+1})$. Moreover, F and $F^{\perp}$ both have the Koszul-Malgrange holomorphicstructures. Let $A_z^{F,F^{\perp}}$ be the (1,0)-part of the second fundamental form of F in $V(\mathbf{C}^{n+1})$. By taking the image of the second fundamental form, we may define the new subbundle of $V(\mathbf{C}^{n+1})$, which is extended to smooth subbundle over S (see [B-P-W]).

Now, starting from L_0, we may have the harmonic sequence $L_0 \to L_1 \to \cdots L_{r-1} \to R$, where $L_i = \mathrm{Im} A_z^{L_{i-1}, L_{i-1}^{\perp}}$ for $i = 1, \cdots, r-1$ and $R = V(\mathbf{C}^{n+1}) \ominus (\bigoplus_{i=0}^{r-1} L_i)$. This situation means that each of $L_0, L_1, \cdots, L_{r-1}$ and R are orthogonal to each other with respect to the Hermitian metric on $V(\mathbf{C}^{n+1})$. In this case, we say that φ has ∂'-isotropy order r. From the definition of harmonic sequence, it is always true that $r \geq 1$. For notational simplicity, set $L_r = R$. Set $G = SU(n+1)$. Fix any point $p \in S$ and define $Q \in G$ by

$$Q = \zeta^i \quad \text{on} \quad (L_i)_p \quad \text{for } i = 0, \cdots, r$$

where $\zeta = \exp(2\pi i / r+1)$. Then, $\tau = AdQ$ is an order $(r+1)$-automorphism of G and the identity component of its fixed set is $S(U(1) \times \cdots \times U(1) \times U(n+1-r))$, which we denote by K. Hence, we define a map $\psi : S \longrightarrow N = G/K$ by

$$\psi(q) = ((L_0)_q, (L_1)_q, \cdots, (L_r)_q), \quad \text{for } q \in S$$

Choose the base point $o = \psi(p)$. The complexification $g^{\mathbf{C}}$ of Lie algebra g of G is decomposed into the eigenspaces of τ:

$$g^{\mathbf{C}} = \sum_{j \in \mathbf{Z}_{r+1}} g_j$$

where

$$m^{\mathbf{C}} = \sum_{j=1}^{r} g_j, \quad k^{\mathbf{C}} = g_0$$

For $x = g \cdot o \in N$, define $\hat{\tau} : N \longrightarrow N$ by $\hat{\tau}(g \cdot o) = \tau(g) \cdot o$. Define $\hat{\tau}_x : N \longrightarrow N$ by $\hat{\tau}_x = g \circ \hat{\tau} \circ g^{-1}$. We may use the Killing form of g to equip N with a metric for which each of the $\hat{\tau}_x$ is an isometry so that N has the structure of an $(r+1)$-symmetric space([K]). Let $[g_i]$ be a subbundle of $N \times g$ of which the fibre at $x \in N$ is given by $[g_i]_x = Adgg_i$. Then, $[g_i]_x$ is ζ^i-eigenspace of $d\hat{\tau}_x$. For $x = g \cdot o \in N$, the map $g \longrightarrow T_xN$ given by

$$\xi \longrightarrow \frac{d}{dt}|_{t=0}\, expt\xi \cdot x$$

restricts to an isomorphism $Adgm \cong T_xN$. The inverse map $\beta_x : T_xN \longrightarrow Adgm \subset g$ may be viewed as a g-valued 1-form β on N, which is called Maurer-Cartan form for N (see [B-R]).

Definition. A map $\psi : S \longrightarrow G/K$ of a Riemann surface is called *primitive* if $\psi^*\beta(\partial/\partial z)$ takes values in $[g_1]$, or equivalently $\alpha'_m(\partial/\partial z)$ takes values in g_1 for any framing $F : S \longrightarrow G$.

Lemma 1.1. *$A_z^{(L_i)_p,(L_{i+1})_p}$ is g_1-valued for $i = 0, \cdots, r$, where $L_{r+1} = L_0$. Moreover, ψ is a primitive map.*

Proof. Let f_i be a local section of L_i for $i = 0, \cdots, r$. If we set $A_z^{L_i, L_{i+1}}(f_i) = a_{i,i+1} f_{i+1}$, then we have, at p

$$\begin{aligned}(AdQ(A_z^{L_i,L_{i+1}}))Qf_i &= Q(a_{i,i+1}f_{i+1}) \\ &= \zeta^{i+1} A_z^{L_i,L_{i+1}}\end{aligned}$$

Therefore, we see that $AdQ(A_z^{L_i,L_{i+1}}) = \zeta A_z^{L_i,L_{i+1}}$. Next, we show the second assertion. From [B-R], we see that $\psi^*\beta(\partial/\partial z)$ is the sum of the second fundamental forms of L_i 's $(i = 0, \cdots, r)$. Then, from the construction of harmonic sequence, it follows that

$$\psi^*\beta(\partial/\partial z) = \sum_{i=0}^{r} A_z^{L_i,L_{i+1}} \tag{1.1}$$

where $L_{r+1} = L_0$ for convention. Since $\psi^*\beta_p = \alpha_m$, it follows from (1.1) that ψ is a primitive map. q.e.d.

It remains to show that α'_m is a semisimple element on a dense subset of S.

2. Non-conformal harmonic maps into $\mathbf{C}P^n$.

Let $\hat{f}_0$ be a local non-zero holomorphic section of L_0 and $(\hat{f}_1, \cdots, \hat{f}_n)$ a local field of unitary frames of $L_1 = L_0^{\perp}$. If we set $f_i = det(\hat{f}_0, \cdots, \hat{f}_n)^{-\frac{1}{n+1}} \hat{f}_i$ for $i = 0, \cdots, n$, then $det(f_0, \cdots, f_n) = 1$. Moreover, we see that $\mid f_0 \mid \cdots \mid f_n \mid = 1$. Now, define F_i by $F_i = \exp(-w_i) f_i$ for $i = 0, \cdots, n$, where $w_i = \log \mid f_i \mid$. Then, $F = (F_0, F_1, \cdots, F_n)$ is $SU(n+1)$-valued locally defined function on S. Let $g = su(n+1)$ be a Lie algebra of $SU(n+1)$. Then, $g^{\mathbf{C}} = sl(n+1, \mathbf{C})$ is decomposed as

$$g^{\mathbf{C}} = g_0 \oplus g_1 \tag{2.1}$$

where g_i is the ζ^i-eigenspace of τ for $i = 0, 1$. We see that g_1 is isomorphic to the complexification of the tangent space at the base point and that $A_z^{L_0,L_0^{\perp}}$ and $A_z^{L_0^{\perp},L_0}$ are g_1-valued (c.f. Lemma 1.1). Write

$$A_z^{L_0,L_0^{\perp}}(f_0) = \sum_{i=1}^{n} a_i f_i, \quad A_{\overline{z}}^{L_0,L_0^{\perp}}(f_0) = \sum_{i=1}^{n} b_i f_i \tag{2.2}$$

Let $g = m + k$ be a Cartan decomposition. Then, $m^{\mathbf{C}} = g_1$. Let $\alpha'_m(\partial/\partial z)$ be a m-part of $F^{-1}(\partial F/\partial z)$. Then, for $j = 1, \cdots, n$

$$(F^{-1}(\partial F/\partial z))_{j,0} = < A_z^{L_0,L_0^{\perp}}(F_0), \overline{F_j} > = a_j w_{j,0} \ ,$$

$$(F^{-1}(\partial F/\partial z))_{0,j} = -\overline{(F^{-1}(\partial F/\partial \overline{z}))_{j,0}} = -\overline{< A_{\overline{z}}^{L_0,L_0^{\perp}}(F_0), \overline{F_j} >} = -\overline{b_j} w_{j,0} \ ,$$

where $w_{j,0} = \exp(w_j - w_0)$. Thus, we obtain

$$\alpha'_m(\partial/\partial z) = \begin{pmatrix} 0 & -\overline{\mathbf{b}} \\ {}^t\mathbf{a} & \mathbf{0}_n \end{pmatrix}$$

where $\mathbf{a} = (a_1 w_{1,0}, a_2 w_{2,0}, \cdots, a_n w_{n,0})$, $\mathbf{b} = (b_1 w_{1,0}, b_2 w_{2,0}, \cdots, b_n w_{n,0})$ and $\mathbf{0}_n$ is an $n \times n$-zero matrix. If we write $< \mathbf{a}, \overline{\mathbf{b}} > = r \cdot \exp(i\theta)$ for non-zero real numbers r, θ, then the eigenvalues of $\alpha'_m(\partial/\partial z)$ are 0 (with multiplicity $n-1$) and $\pm\sqrt{ri}\exp(i\theta/2)$. If we set $A = A_z^{L_0^{\perp},L_0} \circ A_z^{L_0,L_0^{\perp}}$, then we see that $traceA = < \mathbf{a}, \overline{\mathbf{b}} >$. Therefore, $\varphi : S \longrightarrow \mathbf{C}P^n$ is non-conformal at $x \in S$ if and only if $\alpha'_m(\partial/\partial z)$ is semisimple at $x \in S$. If S is a two-torus, $traceA$ is a constant, hence we have proved

Proposition 2.1. *For any non-conformal harmonic map* $T^2 \longrightarrow \mathbf{C}P^n$, $\alpha_m'(\partial/\partial z)$ *is a semisimple element.*

Thus, we have

Theorem 2.2 [B-F-P-P]. *Any non-conformal harmonic map from* T^2 *into* $\mathbf{C}P^n$ *is of finite type.*

3. Weakly conformal harmonic map $S \longrightarrow \mathbf{C}P^n$.

Let $\varphi : S \longrightarrow \mathbf{C}P^n$ be a weakly conformal harmonic map with isotropy order $r(\geq 1)$.

Let $\hat{f}_i$ be a local non-zero holomorphic section of L_i for $i = 0, \cdots, r-1$ and $(\hat{f}_r, \cdots, \hat{f}_n)$ be a local field of unitary frame for L_r. We may suppose, without loss of generality, that

$$A_z^{L_i, L_{i+1}}(f_i) = f_{i+1} \quad \text{for} \;\; i = 0, \cdots, r-2.$$

Set $\hat{f} = (\hat{f}_0, \hat{f}_1, \cdots, \hat{f}_n)$. We reset $f_i = (det \hat{f})^{-\frac{1}{n+1}} \hat{f}_i$ for $i = 0, \cdots, n$. For $i = 0, \cdots, n$, define F_i by $F_i = \exp(-w_i) f_i$, where $w_i = \log \mid f_i \mid$. Then, $F = (F_0, F_1, \cdots, F_n)$ is $SU(n+1)$-valued. Set

$$A_z^{L_{r-1}, L_r}(f_{r-1}) = \sum_{j=r}^{n} a_j f_j \;, \quad A_{\overline{z}}^{L_0, L_r}(f_0) = \sum_{j=r}^{n} b_j f_j \;. \tag{3.1}$$

Then, we have

$$(F^{-1} \partial F / \partial z))_{j,0} = < \partial F_0 / \partial z, \overline{F_j} > = \exp(w_1 - w_0) \delta_{1j}, \qquad (j \geq 1),$$

$$\begin{aligned} (F^{-1} \partial F / \partial z))_{i,j} &= < \partial F_j / \partial z, \overline{F_i} > \\ &= < A_z^{L_j, L_{j+1}}(F_j), \overline{F_i} > = \exp(w_{j+1} - w_j) \delta_{i,j+1}, \;\; (0 \Box i \neq j \Box r-1), \end{aligned}$$

$$\begin{aligned} (F^{-1} \partial F / \partial z))_{i,j} &= < A_z^{L_r, L_r^{\perp}}(F_j), \overline{F_i} > = -\overline{< \overline{F_j}, A_{\overline{z}}^{L_0, L_r}(F_i) >} \\ &= -\overline{b_j} \exp(w_j - w_0) \delta_{i,0} \;\; (\mathrm{mod} \;\; k^{\mathbf{C}}), \;\; (0 \Box i \Box r-1, r \Box j \Box n), \end{aligned}$$

$$\begin{aligned} (F^{-1} \partial F / \partial z))_{i,j} &= < A_z^{L_j, L_{j+1}}(F_j), \overline{F_i} > \\ &= a_i \exp(w_i - w_j) \delta_{j,r-1} \;\; (\mathrm{mod} \;\; k^{\mathbf{C}}), \;\; (r \Box i \Box n, 0 \Box j \Box r-1), \end{aligned}$$

$$(F^{-1} \partial F / \partial z))_{i,j} = 0 \;\; (\mathrm{mod} \;\; k^{\mathbf{C}}) \;\; \text{for} \; r \Box i \neq j \Box n.$$

Therefore, we have

$$
(3.2) \qquad \alpha'_m(\partial/\partial z) = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 0 & -\overline{\mathbf{b}} \\ w_{1,0} & 0 & \cdots\cdots & 0 & 0 & \mathbf{0}_{1,n-r+1} \\ 0 & w_{2,1} & \cdots\cdots & 0 & 0 & \mathbf{0}_{1,n-r+1} \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & w_{r-1,r-2} & 0 & \mathbf{0}_{1,n-r+1} \\ & & \mathbf{0}_{n-r+1,r-1} & & {}^t\mathbf{a} & \mathbf{0}_{n-r+1,n-r+1} \end{pmatrix}
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Present adress :
Department of Mathematics
School of Medicine
Nihon University
Itabashi, Tokyo 173
Japan
e-mail adress : h01217@sinet.ad.jp