PLURIHARMONIC MAPS FROM TORI INTO PROJECTIVE UNITARY GROUPS AND CONFIGURATIONS OF POINTS ON RATIONAL OR ELLIPTIC CURVES

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Abstract. In [19], McIntosh proved that some class of pluriharmonic maps from complex vector spaces into projective unitary groups correspond to maps constructed from triplets $(X, \pi, \mathcal{L})$, consisting of auxiliary Riemann surfaces X, and rational functions π and line bundles $\mathcal{L}$ on X. Such triplet is called a spectral data. McIntosh thus realized the moduli space of such pluriharmonic tori in projective unitary groups as a subset of the moduli space of these spectral data.

Therefore it seems natural to ask the following: *Which spectral data corresponds to a pluriharmonic torus in a projective unitary group?*

In this paper, we give a partial answer to this problem. More precisely, we prove a criterion on the periodicity of pluriharmonic maps constructed from the spectral data whose spectral curves are smooth rational or elliptic curves.

Contents

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1. Introduction

In [17], McIntosh proved that every non-isotropic harmonic torus in a complex projective space corresponds to a map constructed from a triplet $(X, \pi, \mathcal{L})$, consisting of an auxiliary algebraic curve X, and a rational function π and a line bundle $\mathcal{L}$ on X. Such triplet is called a spectral data. In [19], McIntosh also constructed new generalized spectral data which produce pluriharmonic maps from complex vector spaces into complex Grassmann manifolds or projective unitary groups.

Therefore it seems natural to ask the following: *Which spectral data correspond to harmonic tori in complex Grassmann manifolds or projective unitary groups?*

In this paper, we give a partial answer to this problem. More precisely, we prove a criterion on the periodicity of harmonic maps constructed from the spectral data whose spectral curves are smooth rational or elliptic curves.

Before describing the plan of this paper, we now review briefly McIntosh's results and state our main theorems.

McIntosh [19] has constructed a significant correspondence between the following two spaces: the space of pluriharmonic maps $\psi\colon \mathbb{C}^k \to Gr_{k,n+1}$ or $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ of some class, up to isometries, and that of triplets $(X, \pi, \mathcal{L})$ consisting of a compact Riemann surface X(which we call the spectral curve for ψ), a meromorphic function π on X and a line bundle $\mathcal{L}$ over X, which are required to satisfy certain conditions.

This correspondence yields a pluriharmonic map from a spectral data in the following fashion. Take a spectral data $(X, \pi, \mathcal{L})$. On the Jacobian variety $J(X)$ of the spectral curve X, we consider a $l(= k$ or $n+1)$-dimensional linear flow $L\colon \mathbb{C}^l \to J(X), z \mapsto L(z)$. Then we know that each line bundle contained in this flow has the following properties. Denoting by $H^0(X, \mathcal{L} \otimes L(z))$ the space of global holomorphic sections of $\mathcal{L} \otimes L(z)$, we see that the dimension of $H^0(X, \mathcal{L} \otimes L(z))$ is $n+1$ if the degree of π is $n+1$. Let R be the ramification divisor of π. Then, since $(\mathcal{L} \otimes L(z)) \otimes \overline{\rho_X^*(\mathcal{L} \otimes L(z))}$ is isomorphic to the divisor line bundle $\mathcal{O}_X(R)$, each line bundle $\mathcal{L} \otimes L(z)$ has a natural bilinear form h via a trace map $H^0(X, \mathcal{O}_X(R)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}$, which is induced from π. Thus we obtain a vector bundle W of rank $n+1$ over $\mathbb{C}^l$ with the fiber metric h, where the fiber of W at $z \in \mathbb{C}^l$ is given by $H^0(X, \mathcal{L} \otimes L(z))$.

Finally, we construct pluriharmonic maps by using ideal sheaves and connections of W.

In connection with the periodicity, McIntosh observed that pluriharmonic maps associated to the above spectral data $(X, \pi, \mathcal{L})$ has a period v if a certain homomorphism from $\mathbb{C}^l$ to a generalized Jacobian $J(X_{\mathfrak{o}})$ has a period v. However, it is generally hard to compute this homomorphism. In this paper, we compute this homomorphism explicitly.

Now we summarize the content of each section.

In Section 2, we recall the definition of the spectral data, and review, with a slight improvement, McIntosh's construction of pluriharmonic maps in terms of these spectral data.

In Section 3, all spectral data with the smooth rational spectral curve are classified (Theorems 3.9 and 3.10), and corresponding pluriharmonic maps are explicitly constructed (Theorems 3.11 and 3.16). Moreover, we give a sufficient condition for a constructed pluriharmonic map to be periodic (Theorems 3.13 and 3.18).

In Section 4, the proofs of Theorems 3.11 and 3.16 are given. Section 5 is devoted to proving Theorems 3.13 and 3.18.

2. Construction of pluriharmonic maps into projective unitary groups from spectral data

2.1. **Spectral data.** Let $\mathbb{P}^1$ be the smooth rational curve and λ an affine coordinate on it. Let ρ be an anti-holomorphic involution on $\mathbb{P}^1$ defined by $\lambda \mapsto 1/\overline{\lambda}$. Then the fixed point set of ρ consists of the equator S^1 defined by $\{\lambda \in \mathbb{P}^1 \mid |\lambda| = 1\}$.

First we recall the definition of a spectral data introduced by McIntosh (cf. [19]).

Definition 2.1. A spectral data is a triplet $(X, \pi, \mathcal{L})$ of isomorphism classes which satisfies the following conditions:

(1) X is a complete, connected, algebraic curve of arithmetic genus p, with a real involution ρ_X.
(2) π is a meromorphic function on X of degree $N = n+1$ satisfying $\pi \circ \rho_X = 1/\overline{\pi}$, with $n+1$ zeros $P_1, \dots, P_{n+1}$ and $n+1$ poles $Q_l = \rho_X(P_l)$. The points $P_1, \dots P_{n+1}$ may occur in multiple degree. We regard X as a covering of degree $n+1$ of the rational curve $\mathbb{P}^1$ via π.

(3) $\mathcal{L}$ is a line bundle over X of degree $p+n$ satisfying

$$\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \cong \mathcal{O}_X(R),$$

where R is the ramification divisor for π. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we can find a meromorphic function f on X which satisfies the following conditions:
   (a) The divisor (f) of f is given by $D + \rho_{*X}D - R$ and $\overline{\rho_X^* f} = f$.
   (b) Let $X_{\mathbb{R}}$ be the preimage of S^1 by π. Then f is non-negative on $X_{\mathbb{R}}$.

(4) π has no branch points on S^1 and ρ_X fixes every point of $X_{\mathbb{R}}$.

Two triplets are the same if there exists a biholomorphic map between spectral curves which carries the real structure, the meromorphic function and the isomorphism class of the line bundle each other.

When X is a compact connected Riemann surface, the above definition of spectral data becomes simpler.

Theorem 2.2. *Let X be a compact connected Riemann surface. A triplet $(X, \pi, \mathcal{L})$ is a spectral data if and only if it satisfies the following conditions:*

(1) *X is a compact connected Riemann surface of genus p, with real involution ρ_X. The set $X \setminus X^\rho$ consists of two connected components X^N, X^S, where X^ρ is the fixed points of ρ_X. Moreover, X^ρ decomposes into the disjoint union $X^\rho = \coprod_{i=1}^{\nu(X)} S_i^1$ with $S_i^1 = S^1$, that is, $\nu(X)$ copies of a loop.*

(2) *π is a meromorphic function on X of degree $N = n+1$, which satisfies either that all poles are contained in X^N and all zeros are contained in X^S, or that all poles are contained in X^S and all zeros are contained in X^N. Moreover, π has a point $x \in X^\rho$ such that $|\pi(x)| = 1$ and the principal divisor of π has the form $\sum_{i=1}^{n+1} P_i - \sum_{i=1}^{n+1} Q_i$ with $Q_i = \rho_X(P_i)$.*

(3) *$\mathcal{L}$ is a line bundle over X of degree $p+n$ satisfying*

$$D + \rho_{X*}(D) \cong R, \quad \delta(\mathcal{L}) = 0,$$

where R is the ramification divisor for π, D is a divisor such that $\mathcal{L} \cong \mathcal{O}_X(D)$, and $\delta(\mathcal{L})$ is a number defined as follows:

$$\delta(\mathcal{L}) = \nu(X) - |\sharp\{s_i \in \Lambda \mid g(s_i)/g(s_1) > 0\} - \sharp\{s_i \in \Lambda \mid g(s_i)/g(s_1) < 0\}|,$$

where g is a meromorphic function with the divisor $(g) = D + \rho_{X*}D - R$ and Λ is the set of points $s_1, s_2, \ldots, s_{\nu(X)}$ such that $s_i \in S_i^1$ and $g(s_i) \neq 0, \infty$.

(The proof of this theorem is in [30].)

2.2. **A Hermitian inner product on a space of global holomorphic sections.** From now on, for a Riemann surface X and a sheaf $\mathcal{F}$ on X, we denote by $H^i(X, \mathcal{F})$ and $H^i(Y, \mathcal{F})$ the i-th cohomology of the sheaf of holomorphic sections of $\mathcal{F}$ and its restriction to an open subset Y of X, respectively. We also denote the dimension of $H^i(X, \mathcal{F})$ by $h^i(X, \mathcal{F})$.

Let $(X, \pi, \mathcal{L})$ be a spectral data as in Definition 2.1. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we equip $H^0(X, \mathcal{L})$ with a positive definite Hermitian form h as follows.

For given $u, v \in H^0(X, \mathcal{L})$, we define a rational function $h(u, v)$ on $\mathbb{P}^1$ by

$$h(u,v)(p) = \sum_{x \in \pi^{-1}(p)} f(x)u(x)\overline{(v \circ \rho_X)(x)}, \tag{2.3}$$

where p is a point of $\mathbb{P}^1$. Then it is known that $h(u, v)$ is a constant function and the following holds.

Theorem 2.4. ([18]) *The Hermitian form h is positive definite on $H^0(X, \mathcal{L})$. Moreover, $\pi_*\mathcal{L}$ is a trivial vector bundle of rank $(n + 1)$ over $\mathbb{P}^1$, where $n + 1$ is the degree of π.*

Let η be a point on S^1 and $\pi^{-1}(\eta) = \{\eta_0, \dots, \eta_n\}$, the inverse image of 1 by π, and $\theta_i (0 \leqq i \leqq n)$ a local trivialization for $\mathcal{L}$ over a neighbourhood of η_i. Using these local trivializations, the Hermitian form h in (2.3) has also the following expression. For $u \in H^0(X, \mathcal{L})$, let $u_0, \dots, u_n$ be the complex numbers defined by $u(\eta_i) = u_i\theta_i(\eta_i)$. For $v \in H^0(X, \mathcal{L})$, we define the complex numbers $v_0, \dots, v_n$ in a similar way. Then (2.3) becomes

$$h(u, v) = \sum_{i=0}^{n} a_i u_i \overline{v_i}, \tag{2.5}$$

where $a_0, \dots, a_n$ are positive real numbers depending only on the choice of $\theta_0, \dots, \theta_n$. From the result above we obtain an orthonormal basis $\{\sigma_i\}$ of $H^0(X, \mathcal{L})$ such that

$$\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$$

for $0 \leqq i \leqq n$.

2.3. **Construction of pluriharmonic maps into projective unitary groups.**

Definition 2.6. A spectral data of type Pr_{n+1} is a spectral data $(X, \pi, \mathcal{L})$ as in Theorem 2.2 which satisfies the following conditions: The degree of π is $n + 1$ $(n \geqq 1)$. Moreover π has distinguished $n + 1$ zeros, that is

$$P_i \neq P_j \quad \text{for} \quad i \neq j.$$

Let $(X, \pi, \mathcal{L})$ be a spectral data of type Pr_{n+1}. It will be shown that the corresponding map is a pluriharmonic map from $\mathbb{C}^{n+1}$ to $\mathbb{P}U_{n+1}$.

Let $\pi^{-1}(1) = \{\eta_0, \ldots, \eta_n\}$, the inverse image of 1 by π. Applying the method in Section 2.2 to $\eta = 1$, we obtain an orthonormal basis $\{\sigma_i\}$ of $\mathcal{L}$ such that

$$\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$$

for $0 \leqq i \leqq n$.

Let $\pi^{-1}(-1) = \{\nu_0, \ldots, \nu_n\}$, the inverse image of -1 by π. Applying the method in Section 2.2 to $\eta = -1$, we obtain another orthonormal basis $\{\rho_i\}$ of $\mathcal{L}$ such that

$$\rho_i \in H^0(X, \mathcal{L}(-\nu_0 - \cdots - \nu_{i-1} - \nu_{i+1} - \cdots - \nu_n))$$

for $0 \leqq i \leqq n$.

Next we construct a line bundle $L(z)$ with a complex parameter $z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$. For $1 \leqq l \leqq n+1$, let $U(P_l)$ be a neighbourhood of P_l and $U(Q_l)$ a neighbourhood of Q_l defined by $U(Q_l) = \rho_X(U(P_l))$. For $1 \leqq l \leqq n+1$, let ζ_l be a meromorphic function on $U(P_l) \cup U(Q_l)$ satisfying $\pi = \zeta_l$ and $\zeta_l \circ \rho_X = 1/\bar{\zeta}_l$. We fix an open cover $X_A \cup X_I$ of X, where $X_A = X \setminus \{P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1}\}$ and $X_I = \cup_{l=1}^{n+1}(P_l \cup Q_l)$. Let $L(z)$ be the unique line bundle with local trivializations θ_A^z and θ_I^z over X_A and X_I respectively, such that

$$\theta_I^z = \exp\left(\sum_{l=1}^{n+1}\left(z_k\zeta_l^{-1} - \bar{z}_l\zeta_l\right)\right)\theta_A^z \quad \text{on } X_A \cap X_I. \tag{2.7}$$

We denote by $J(X)_{\mathbb{R}}$ the connected component of the kernel of $J(X) \to J(X), \mathcal{O}_X(D) \mapsto \mathcal{O}_X(D + \rho(D))$, which contains the trivial line bundle.

Let $J(X_{\mathfrak{m}})_{\mathbb{R}}$ be a bundle over $J(X)_{\mathbb{R}}$ whose fiber $J_{\mathfrak{m}}(L)$ at $L \in J(X)_{\mathbb{R}}$ is given by

$$J_{\mathfrak{m}}(L) = \prod_{i=1}^{n}(\mathrm{Hom}\,(L|_{\eta_i},\, L|_{\eta_0}) \setminus \{0\}),$$

and let $J(X_{\mathfrak{n}})_{\mathbb{R}}$ be a bundle over $J(X)_{\mathbb{R}}$ whose fiber $J_{\mathfrak{n}}(L)$ at $L \in J(X)_{\mathbb{R}}$ is given by

$$J_{\mathfrak{n}}(L) = \prod_{i=1}^{n}(\mathrm{Hom}\,(L|_{\nu_i},\, L|_{\nu_0}) \setminus \{0\}).$$

Setting $J_{\mathfrak{o}}(L) = J_{\mathfrak{m}}(L) \times J_{\mathfrak{n}}(L)$ for $L \in J(X)_{\mathbb{R}}$, we get a new bundle

$$J(X_{\mathfrak{o}})_{\mathbb{R}} = \bigcup_{L \in J(X)_{\mathbb{R}}} J_{\mathfrak{o}}(L)$$

over $J(X)_{\mathbb{R}}$. We define a map $\hat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{o}})_{\mathbb{R}}$ by $z \mapsto (L(z), g_1^z, \ldots, g_n^z, h_1^z, \ldots, h_n^z)$, where g_i^z is an element of $\mathrm{Hom}\,(L(z)|_{\nu_i},\, L(z)|_{\nu_0}) \setminus \{0\}$ $(\cong \mathbb{C}^*)$ defined by the condition that g_i^z maps $\theta_A(z)|_{\nu_i}$ to $\theta_A(z)|_{\nu_0}$ and h_i^z an element of $\mathrm{Hom}\,(L(z)|_{\eta_i},\, L(z)|_{\eta_0}) \setminus \{0\}$ $(\cong$

$\mathbb{C}^*$) defined by the condition h_i^z maps $\theta_A(z)|_{\eta_i}$ to $\theta_A(z)|_{\eta_0}$. Then we see that $\hat{L}$ is a homomorphism from the additive group $\mathbb{C}^{n+1}$ to $J(X_{\mathfrak{o}})_{\mathbb{R}}$.

For $0 \leqq i \leqq n$, let $\mathcal{L}_i$ be an ideal sheaf of $\mathcal{L}$ defined by $\mathcal{L}_i = \mathcal{L}(\nu_i - \sum_{j=0}^n \nu_j)$. Then it is known that $H^0(X, \mathcal{L}_i \otimes L)$ is a 1-dimensional complex vector space for $0 \leqq i \leqq n$ and $L \in J(X)_{\mathbb{R}}$. Take any nonzero global section $t_i(L)$ of $\mathcal{L}_i \otimes L$ for $0 \leqq i \leqq n$. Next we define a map $pr\colon J(X_{\mathfrak{o}})_{\mathbb{R}} \to \mathbb{P}U_{n+1}$ by $(L, g_1, \dots, g_n, h_1, \dots, h_n) \mapsto [(\psi_{i,j})]$, where $\psi_{i,j}$ is given by

$$\frac{1}{g_i\left(t_i/\rho_i|_{\nu_i}\right)} \otimes h_j\left(t_i/\sigma_j|_{\eta_j}\right) \quad \in L^{-1}|_{\nu_0} \otimes L|_{\eta_0}.$$

Here we use the identification

$$\mathfrak{gl}(n+1) \cong \mathfrak{gl}(n+1) \otimes L^{-1}|_{\nu_0} \otimes L|_{\eta_0}, \quad (M_{ij}) \mapsto \left(M_{ij}(1/\rho_0)|_{\nu_0} \otimes \sigma_0|_{\eta_0}\right).$$

Let $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$, $z \mapsto [(\psi_{i,j}(z))]$ be a map defined as the composition $pr \circ \hat{L}$. Then it is known that ψ is a pluriharmonic map corresponding to the spectral data $(X, \pi, \mathcal{L})$.

Theorem 2.8. *Let $\phi_i(z, x)$ be a section of $\mathcal{L}$ over X_A such that $\phi_i(z, x)\theta_A(z)$ can be extended to a global section of $\mathcal{L}_i \otimes L(z)$ and $(\phi_i(z, x) - \rho_i)|_{\nu_i} = 0$. Then we get the another expression for $\psi_{i,j}(z)$ (cf.* [30]*)*

$$\psi_{i,j}(z) = \frac{\phi_i(z, \eta_j)}{\sigma_j(\eta_j)} \quad \text{for } 0 \leqq i, j \leqq n. \tag{2.9}$$

Before closing this section, we prove the following lemma for later use.

Lemma 2.10. *Given a function $\phi(x)$ on X, let U and V be neighbourhoods of the set of the points $\{x_1, \dots x_p\}$ which satisfy the following conditions:*

1. *$\{x_1, \dots, x_p\} \subset U \subset V$.*
2. *$\phi(x)$ is a holomorphic section of $\mathcal{O}_X(M)$ on $X \setminus U$, where M is a divisor on $X \setminus V$.*
3. *$\phi(x)\exp(t)$ extends to V as a holomorphic section of $\mathcal{O}_X(N)$ on V, where N is a divisor on U and t is a holomorphic function on $V \setminus \{x_1, \dots x_p\}$.*

Moreover, let L be the unique line bundle with local trivializations θ_A and θ_I over $X \setminus \{x_1, \dots x_p\}$ and V respectively, such that

$$\theta_I = \exp(t)\,\theta_A \quad \text{on } V \setminus \{x_1, \dots x_p\}. \tag{2.11}$$

Then $\phi(x) \otimes \theta_A$ extends to X as a holomorphic section of $\mathcal{F} \otimes L$, where $\mathcal{F} \cong \mathcal{O}_X(M + N)$.

Proof. From the condition (2), $\phi(x) \otimes \theta_A$ clearly belongs to $H^0(X \setminus U, \mathcal{O}_X(M) \otimes L) = H^0(X \setminus U, \mathcal{F} \otimes L)$. It suffices to show that $\phi(x) \otimes \theta_A$ belongs to $H^0(V, \mathcal{O}_X(N) \otimes L) = H^0(V, \mathcal{F} \otimes L)$. By using (2.11), we see that $\phi(x) \otimes \theta_A = \phi(x) \exp(-t) \otimes \theta_I$ on V. On the other hand, from the condition (3) it follows that $\phi(x)\exp(t)$ is an element of $H^0(V, \mathcal{F})$ and hence $\phi(x) \otimes \theta_A$ belongs to $H^0(V, \mathcal{F} \otimes L)$. Thus $\phi(x) \otimes \theta_A$ is a global holomorphic section of $\mathcal{F} \otimes L$ on X. □

3. Main results

3.1. **Jacobi's theta functions and Weierstrass' zeta functions.** C. G. J. Jacobi introduced four functions θ_1, θ_2, θ_3 and θ_4 of variables $p(u) = \exp(\pi\sqrt{-1}u)$ and $q = \exp(\pi\sqrt{-1}\tau)$, where u is the usual covering coordinate of an elliptic curve $X = \mathbb{C}/\mathbb{L}$ and τ stands for its period ratio with familiar standardization that the imaginary part $\mathrm{Im}\tau$ of τ is positive. If we take $\mathbb{L}$ to be $\mathbb{Z} \oplus \tau\mathbb{Z}$ for simplicity, then these Jacobi's theta functions are given as follows:

$$\begin{aligned}
\theta_1(u) &= \theta_1(u|\tau) = \sqrt{-1}\sum(-1)^n p^{2n-1} q^{(n-1/2)^2},\\
\theta_2(u) &= \theta_2(u|\tau) = \sum p^{2n-1} q^{(n-1/2)^2},\\
\theta_3(u) &= \theta_3(u|\tau) = \sum p^{2n} q^{n^2},\\
\theta_4(u) &= \theta_4(u|\tau) = \sum(-1)^n p^{2n} q^{n^2}.
\end{aligned}$$

Here the sums are taken over $n \in \mathbb{Z}$. Under the addition of half-periods, these functions transform according to the following table.

	$u+1/2$	$u+\tau/2$	$u+1/2+\tau/2$	$u+1$	$u+\tau$	$u+1+\tau$
θ_1	θ_2	$\sqrt{-1}a\theta_4$	$a\theta_3$	$-\theta_1$	$-b\theta_1$	$b\theta_1$
θ_2	$-\theta_1$	$a\theta_3$	$-\sqrt{-1}a\theta_4$	$-\theta_2$	$b\theta_2$	$-b\theta_2$
θ_3	θ_4	$a\theta_2$	$\sqrt{-1}a\theta_1$	θ_3	$b\theta_3$	$b\theta_3$
θ_4	θ_3	$\sqrt{-1}a\theta_1$	$a\theta_2$	θ_4	$-b\theta_4$	$-b\theta_4$

For example, we have the transformation rules

$$\theta_1(u+\tau) = -b(u)\theta_1(u), \tag{3.1}$$

$$\theta_1(u+1/2) = \theta_2(u), \tag{3.2}$$

$$\theta_1(u+\tau/2) = -\sqrt{-1}a(u)\theta_4(u), \tag{3.3}$$

$$\theta_3(u+\tau/2) = a(u)\theta_2(u), \tag{3.4}$$

$$\theta_4(u+1/2) = \theta_3(u), \tag{3.5}$$

where $a(u) = p(u)^{-1}q^{-1/4}$ and $b(u) = p(u)^{-2}q^{-1}$. Special values of these functions are obtained as follows:

$$\begin{aligned}\lim_{t\to\infty} q^{-1/4}\frac{\partial\theta_1}{\partial u}(0|\sqrt{-1}t) = 2\pi, \quad \lim_{t\to\infty} q^{-1/4}\theta_2(0|\sqrt{-1}t) = 2,\\ \lim_{t\to\infty}\theta_3(0|\sqrt{-1}t) = 1, \quad \lim_{t\to\infty}\theta_4(0|\sqrt{-1}t) = 1.\end{aligned} \tag{3.6}$$

On the other hand, Weierstrass' zeta function ζ_w is defined by

$$\zeta_w(u) = \zeta_{w,\tau}(u) = \frac{1}{u} + \sum_{\omega\in\mathbb{L}\setminus(0,0)}\left\{\frac{1}{(u-\omega)} + \frac{u}{\omega^2} + \frac{1}{\omega}\right\}. \tag{3.7}$$

Note that these functions have the following properties. θ_1 is a odd function. θ_2, θ_3 and θ_4 are even functions. Concerning ζ_w, there exist complex numbers $A = A_\tau$ and $B = B_\tau$ depending only on τ such that

$$\zeta_w(u+1) - \zeta_w(u) = A, \quad \zeta_w(u+\tau) - \zeta_w(u) = B, \quad A\tau - B = 2\pi\sqrt{-1}. \tag{3.8}$$

Moreover, if τ is pure imaginary, we have $\overline{\theta_1(u)} = \theta_1(\overline{u})$, $\overline{\zeta_w(u)} = \zeta_w(\overline{u})$, $\overline{A} = A$ and $\overline{B} = -B$.

For further details and formulas regarding these functions, we refer the reader to McKean and Moll [23, Chapter 3].

Our main theorems which refine the correspondence proved by McIntosh may be stated in the following.

3.2. **Spectral data with smooth rational or elliptic spectral curves.** To describe spectral data with the rational spectral curve, we prepare the following functions. For any point P on $\mathbb{P}^1$, we define a meromorphic function $\Lambda^P(\lambda)$ by

$$\Lambda^P(\lambda) = \begin{cases} \dfrac{1}{\overline{\overline{P}}}\dfrac{(\lambda - P)}{(\lambda - 1/\bar{P})} & \text{for } \lambda(P)\in\mathbb{C}\setminus\{0\} \\ 1/\lambda & \text{for } \lambda(P) = \infty \\ \lambda & \text{for } \lambda(P) = 0. \end{cases}$$

Theorem 3.9. *Let X be the smooth rational curve. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if it is isomorphic to the following:*

(1) *(X, ρ_X) is real isomorphic to $(\mathbb{P}^1, \rho)$. By the affine coordinate λ, π is expressed as*

$$\pi(\lambda) = \alpha_0^{-1}\prod_{j=1}^{n+1}\Lambda^{P_j}(\lambda), \quad \alpha_0 = \prod_{j=1}^{n+1}\Lambda^{Q_j}(1).$$

Here $P_j \in X^S = \{\lambda\in X \mid |\lambda| < 1\}$ and $Q_j = 1/\overline{P_j}$ for any $1 \leqq j \leqq n+1$.

(2) *$\mathcal{L}$ is a line bundle of degree n.*

For the above spectral data $(\mathbb{P}^1, \pi, \mathcal{L})$ and any point P on $\mathbb{P}^1$, we define a meromorphic function $\sigma^P(\lambda)$ by

$$\sigma^P(\lambda) = \begin{cases} \dfrac{\lambda - P}{\lambda - P_{fix}} & \text{for } \lambda(P) \in \mathbb{C} \\ 1/(\lambda - P_{fix}) & \text{for } \lambda(P) = \infty \end{cases}$$

Here P_{fix} is a fixed point on X such that $P_{fix} \notin \cup_{j=0}^n (P_j \cup Q_j \cup S^1 \cup R_j)$. Moreover $\{\eta_0, \ldots, \eta_n\}$ is the inverse image $\pi^{-1}(1)$ of 1 by π and $R_+ = \sum_{j=1}^n R_j$ is a divisor given by the intersection of X^S with R, that is, $R_+ = X^S \cap R$.

Next, we consider the case of a smooth elliptic spectral curve $X = X_\tau$. Let us denote by $\mathrm{Pic}^d(X)$ and $J(X)$ the set of line bundles on X of degree d and the Jacobian of X, respectively. Note that $J(X)$ can be identified with $X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. We then define a biholomorphic map $J\colon \mathrm{Pic}^0(X) \to J(X)$ by $J(L) = \sum_{j=1}^k (P_j - Q_j) \pmod{\mathbb{Z} \oplus \mathbb{Z}\tau}$, provided that $L \in \mathrm{Pic}^0(X)$ is expressed as a divisor line bundle $\mathcal{O}_X(\sum_{j=1}^k (P_j - Q_j))$. For any point P on $\mathbb{C}$, it is more convenient to define a holomorphic function $\theta(P, u)$ on $\mathbb{C}$ by

$$\theta(P, u) = \theta_1(u - P) = \theta_1(u - P|\tau)$$

where θ_1 is the Jacobi's theta function(cf. section 3.1).

In connection with spectral data with smooth elliptic spectral curves, we shall see the followings:

Theorem 3.10. *Let X be a smooth elliptic curve. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if it is isomorphic to the following:*

(1) *X is an elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$, where τ is a pure imaginary number $\sqrt{-1}t$ with $t > 0$. ρ_X is an anti-holomorphic involution induced by the usual conjugation of $\mathbb{C}$. Regarded as a doubly periodic meromorphic function on $\mathbb{C}$, π is expressed as*

$$\pi(u) = C \frac{\prod_{j=1}^n \theta(P_j, u) \cdot \theta(P_{n+1} + W, u)}{\prod_{j=1}^{n+1} \theta(Q_j, u)}$$

Here $P_i \in X^S = \{x \in X \mid 0 < \mathrm{Im}\, x < \mathrm{Im}\, \tau/2 \pmod{\mathrm{Im}\, \tau\mathbb{Z}}\}$ and $Q_i = \overline{P_i} \pmod{\mathbb{Z} \oplus \mathbb{Z}\tau}$ for any $0 \leqq i \leqq n+1$; $W = \sum_{i=1}^{n+1} P_i - \sum_{i=1}^{n+1} Q_i$; W belongs to $\mathbb{Z} \oplus \mathbb{Z}\tau$; and C is the unique constant such that $\pi(0) = 1$.

(2) *Let $r : \mathrm{Pic}^{n+1}(X) \to \mathrm{Pic}^0(X)$ be a map defined by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_X(-R_+)$, where $R_+ = \sum_{j=0}^n R_j$ is a divisor of degree $n+1$ given by the intersection of X^S with R, that is, $R_+ = X^S \cap R$. Then, $\mathcal{L}$ is an element of the inverse image of $\left(\mathbb{Z} \oplus \sqrt{-1}\mathbb{R}\right) / (\mathbb{Z} \oplus \tau\mathbb{Z})$ by the composition $J \circ r$.*

3.3. **Pluriharmonic maps into projective unitary groups.** Our main theorems concerning projective unitary groups, which refine the correspondence proved by McIntosh, may be stated as follows. (See Section 2.3 for the detail of this correspondence.)

Theorem 3.11. *Let $(X,\pi,\mathcal{L})$ be of type Pr_{n+1} among spectral data as in Theorem 3.9. Choosing a complex coordinate on the source suitably, the pluriharmonic map $\Psi\colon \mathbb{C}^{n+1}\to \mathbb{P}U_{n+1}$ corresponding to the above spectral data is given by*

$$z\mapsto [(\Psi_{i,j}(z))],$$

where

(3.12)

$$\Psi_{i,j}(z)=\exp\left(\sum_{j=1}^{n+1}\Lambda^{Q_j}(\eta_i)z_j-\sum_{j=1}^{n+1}\Lambda^{P_j}(\eta_i)\overline{z_j}\right)\cdot\frac{\prod_{k=0}^{i-1}\sigma^{\nu_k}(\eta_j)\cdot\prod_{k=i+1}^{n}\sigma^{\nu_k}(\eta_j)\prod_{k=1}^{n}\sigma^{R_k}(\nu_i)}{\prod_{k=1}^{n}\sigma^{R_k}(\eta_j)\prod_{k=1}^{i-1}\sigma^{\nu_k}(\nu_i)\prod_{k=i+1}^{n}\sigma^{\nu_k}(\nu_i)},$$

where $\{\eta_0,\dots,\eta_n\}$ is the inverse image $\pi^{-1}(1)$ of 1 by π and $\{\nu_0,\dots,\nu_n\}$ is the inverse image $\pi^{-1}(-1)$ of -1 by π.

Furthermore we obtain the following

Theorem 3.13. *Let $\Psi\colon\mathbb{C}^{n+1}\to\mathbb{P}U_{n+1}$ be the pluriharmonic map in Theorem 3.11. Then Ψ is a lift of a map from $\mathbb{C}^{n+1}/\Gamma$ with a p-dimensinal lattice $\Gamma=\oplus_{l=1}^{p}\mathbb{Z}v_p\subset\mathbb{C}^{n+1}$ if the set $V=\bigcap_{1\leqq i\leqq 2n}V_i$ contains Γ, where $V_1,\dots,V_{2n}$ are the sets defined by*

$$V_i=\{z\in\mathbb{C}^k\mid f_i(z)=\sum_{l=1}^{k}\beta_{i,l}z_l\in\pi\left(\mathbb{R}\oplus\sqrt{-1}\mathbb{Z}\right)\}\tag{3.14}$$

where $f_0,\dots f_{2n}$ are linear holomorphic functions on $\mathbb{C}^{n+1}$ defined by

$$\beta_{i,l}=\begin{cases}\Lambda^{Q_l}(\nu_i) & (1\leqq i\leqq n)\\ \Lambda^{Q_l}(\eta_i) & (n+1\leqq i\leqq 2n).\end{cases}$$

Corollary 3.15. *Let $(X,\pi,\mathcal{L})$ be a spectral data in Theorem 3.11 such that the degree of π is 2. Then the corresponding pluriharmonic map $\Psi\colon\mathbb{C}\to\mathbb{P}U_2$ in Theorem 3.11 is always doubly periodic with periods v_1, v_2, where v_1 and v_2 are complex numbers in the set*

$$\mathbb{Z}v_+\oplus\mathbb{Z}v_-=\mathbb{Z}\,\pi\left(\beta_1\mathrm{Im}(\beta_2/\beta_1)\right)^{-1}\oplus\mathbb{Z}\,\pi\left(\beta_2\mathrm{Im}(\beta_1/\beta_2)\right)^{-1}.$$

Here $\beta_1=\Lambda^{Q_1}(\eta_1)-\Lambda^{Q_1}(\eta_0)$ and $\beta_2=\Lambda^{Q_1}(\nu_1)-\Lambda^{Q_1}(\eta_0)$.

Proof. In this case, the set V in Theorem 3.13 reduces to $\mathbb{Z}v_+\oplus\mathbb{Z}v_-$. Hence Corollary 3.15 follows from Theorem 3.13. □

Now we consider the case of a smooth elliptic spectral curve X.

Theorem 3.16. *Let $(X, \pi, \mathcal{L} = \mathcal{O}_X(D))$ be of type Pr_{n+1} among spectral data as in Theorem 3.10. Choosing a complex coordinate on the source suitably, the pluriharmonic map $\Psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ corresponding to the above spectral data is given by*

$$z \mapsto [(\Psi_{i,j}(z))],$$

where $\Psi_{i,j}(z)$ is a function defined by

$$\Psi_{i,j}(z) = \frac{\mu_j}{\widetilde{\mu_i}} \frac{\exp\left(\sum_{k=1}^{n+1}([\zeta_w(\eta_j - P_k) - A\,\eta_j]z_k - [\zeta_w(\eta_j - Q_k) - A\,\eta_j]\overline{z_k})\right)}{\exp\left(\sum_{k=1}^{n+1}([\zeta_w(\nu_i - P_k) - A\,\nu_i]z_k - [\zeta_w(\nu_i - Q_k) - A\,\nu_i]\overline{z_k})\right)}$$

$$\cdot \frac{\prod_{k=0}^{n}\theta(R_k,\nu_i)\prod_{k=0}^{j-1}\theta(\nu_k,\eta_j)\cdot\theta(\widehat{\nu_i} + \sum_{k=1}^{n+1}(z_k - \overline{z_k}),\eta_j)\prod_{k=j+1}^{n}\theta(\nu_k,\eta_j)}{\prod_{k=0}^{n}\theta(R_k,\eta_j)\prod_{k=0}^{j-1}\theta(\nu_k,\nu_i)\cdot\theta(\widehat{\nu_i} + \sum_{k=1}^{n+1}(z_k - \overline{z_k}),\nu_i)\prod_{k=j+1}^{n}\theta(\nu_k,\nu_i).} \tag{3.17}$$

Here $\{\eta_0, \ldots, \eta_n\}$ and $\{\nu_0, \ldots, \nu_n\}$ are the inverse images of 1 and -1 by π respectively, μ_i and $\widetilde{\mu_i}$ are constants given by

$$\mu_i = \exp\left(2\pi\sqrt{-1}(D - R_+)\mathrm{Im}\,\eta_i/t\right), \quad \widetilde{\mu_i} = \exp\left(2\pi\sqrt{-1}(D - R_+)\mathrm{Im}\,\nu_i/t\right).$$

Moreover $\widehat{\nu_i}$ is a constant defined by $\widehat{\nu_i} = D + \nu_i - \sum_{j=0}^{n}\nu_j$ and A is a constant given in the equation (3.8).

Moreover we prove the following

Theorem 3.18. *Let $\Psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ be the pluriharmonic map in Theorem 3.16. Then Ψ is a lift of a map from $\mathbb{C}^{n+1}/\Gamma$ with a p-dimensinal lattice $\Gamma = \oplus_{l=1}^{p}\mathbb{Z}v_p \subset \mathbb{C}^{n+1}$ if the set $V = \bigcap_{0\leqq i\leqq 2n} V_i$ contains Γ, where $V_0, \ldots, V_{2n}$ are the sets defined by*

$$V_i = \{z \in \mathbb{C}^k \mid f_i(z) = \sum_{l=1}^{n+1}\beta_{i,l}z_l \in \pi\left(\mathbb{R}\oplus\sqrt{-1}\mathbb{Z}\right)\} \tag{3.19}$$

where $f_0, \ldots f_{2n}$ are linear holomorphic functions on $\mathbb{C}^{n+1}$ defined by

$$\beta_{i,l} = \begin{cases} ([\zeta_w(\nu_0 - P_l) - \zeta_w(\nu_i - P_l) - B\,(\nu_0 - \nu_i)\tau^{-1}]) & (1 \leqq i \leqq n) \\ ([\zeta_w(\eta_0 - P_l) - \zeta_w(\eta_i - P_l) - B\,(\eta_0 - \eta_i)\tau^{-1}]) & (n+1 \leqq i \leqq 2n) \\ 2\pi/t & (i = 0). \end{cases}$$

4. **Explicit construction of pluriharmonic maps**

In this section we prove Theorems 3.11 and 3.16.

First, we construct special orthonormal bases of certain spaces of global sections of holomorphic line bundles. Let $(X, \pi, \mathcal{L})$ be a spectral data as in Theorem 3.9. We may assume that π, R and $\mathcal{L}$ are of the following form:

$$\pi(\lambda) = \alpha_0^{-1}\prod_{j=1}^{n+1}\Lambda^{P_j}(\lambda), \quad R = D + \rho_X(D), \quad \mathcal{L} = \mathcal{O}_X(D),$$

where α_0 is a constant as in Theorem 3.9 and D is a divisor defined by $D = R \cap X_S = \sum_{i=1}^{n} R_i$.

Next we construct a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{n} R_i)$ following the method explained above. Here we choose $f = 1$ as a meromorphic function on X in Condition (3) of Definition 2.1. For $0 \leqq i \leqq n$, let us denote by σ_i the following element

$$\sigma_i(\lambda) = \frac{\prod_{j=1}^{n} \sigma^{R_j}(\eta_i)}{\prod_{j=0}^{i-1} \sigma^{\eta_j}(\eta_i) \prod_{j=i+1}^{n} \sigma^{\eta_j}(\eta_i)} \frac{\prod_{j=0}^{i-1} \sigma^{\eta_j}(\lambda) \prod_{j=i+1}^{n} \sigma^{\eta_j}(\lambda)}{\prod_{j=1}^{n} \sigma^{R_j}(\lambda)}.$$

Then we see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. Similarly, by replacing η_i by ν_i, we get another orthonormal basis $\{\rho_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\rho_i, \rho_j) = \delta_{ij}$.

Let $\left(X = X_{\sqrt{-1}t}, \pi, \mathcal{O}_X(D) = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i)\right)$ be a spectral data as in Theorem 3.10.

Next we construct a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{d+n+1} E_i - \sum_{i=1}^{d} F_i)$ following the method used in Section 2.2. Here we choose

$$f = \frac{\prod_{j=1}^{k+n+1} \theta(E_j, u)}{\prod_{j=1}^{k} \theta(F_j, u) \prod_{j=0}^{n} \theta(R_j, u)} \cdot \frac{\prod_{j=1}^{k+n+1} \theta(\overline{E_j}, u)}{\prod_{j=1}^{k} \theta(\overline{F_j}, u) \prod_{j=0}^{n} \theta(\overline{R_j}, u)}$$

as a meromorphic function on X in Condition (3) of Definition 2.1. Let μ_i be the constant in Theorem 3.16 and set $\widehat{\eta_i} = \sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i - (\eta_0 + \cdots + \eta_{i-1} + \eta_{i+1} + \cdots + \eta_n)$. Denoting by σ_i the element

$$\mu_i^{-1} \frac{\prod_{j=0}^{n} \theta(R_j, \eta_i) \cdot \prod_{j=1}^{k} \theta(F_j, u) \cdot \prod_{j=0}^{i-1} \theta(\eta_j, u) \cdot \theta(\widehat{\eta_i}, u) \cdot \prod_{j=i+1}^{n} \theta(\eta_j, u)}{\prod_{j=1}^{i-1} \theta(\eta_j, \eta_i) \cdot \theta(\widehat{\eta_i}, \eta_i) \cdot \prod_{j=i+1}^{n} \theta(\eta_j, \eta_i) \cdot \prod_{j=1}^{k+n+1} \theta(E_j, u)},$$

e see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. These are well-defined by the following lemma.

Lemma 4.1. *The above constants $\widehat{\eta_i}$ are not equal to η_i $(\mathrm{mod}\ \mathbb{Z} \oplus \mathbb{Z}\tau)$.*

Proof. If $\widehat{\eta_i} = \eta_i \mod \mathbb{Z} \oplus \mathbb{Z}\tau$, then $h(\sigma_i, \sigma_i) = 0$, which is a contradiction because h is positive definite. □

Similarly, by replacing η_i by ν_i, we get another orthonormal basis $\{\rho_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\rho_i, \rho_j) = \delta_{ij}$.

4.1. **Explicit construction of pluriharmonic maps into projective unitary groups.** Using the results in Section 2.3, let us now construct pluriharmonic maps corresponding to spectral data whose spectral curves are smooth rational curves, and prove Theorem 3.11.

Let $(X, \pi, \mathcal{L})$ be a spectral data as in Theorem 3.11. We may assume that π, R and $\mathcal{L}$ are of the following form:

$$\pi(\lambda) = \alpha_0 \prod_{j=1}^{n+1} \Lambda^{P_j}(\lambda), \quad R = D + \rho_X(D), \quad \mathcal{L} = \mathcal{O}_X(D),$$

where α_0 is a constant as in Theorem 3.9 and D is a divisor defined by $D = R \cap X_S = \sum_{i=1}^{n} R_i$. First we prove the following

Lemma 4.2. *Let $(X, \pi, \mathcal{L})$ be a spectral data as above. Define a function $\psi_i(z, \lambda)$ on X with parameter z by*

$$\psi_i(z,\, \lambda) = \exp\left(\sum_{j=1}^{k}\left(\Lambda^{Q_j}(\lambda)\frac{z_j}{\kappa_j} - \Lambda^{P_j}(\lambda)\overline{\left(\frac{z_j}{\kappa_j}\right)}\right)\right) \cdot \frac{\prod_{j=1}^{n+1}\sigma^{\nu_j}(\lambda)}{\sigma^{\nu_i}(\lambda)\prod_{j=1}^{n}\sigma^{R_j}(\lambda)}. \tag{4.3}$$

Here $\kappa_j = (\partial\zeta_j/\partial\lambda)|_{\lambda=P_j}$ is the value of the differential of the meromorphic function ζ_j as in (2.7) at $\lambda = P_j$. Then $\psi_i(z, \lambda)\theta_A(z)$ is an element of $H^0(X,\, \mathcal{L}_i \otimes L(z))$ for any $z \in \mathbb{C}^{n+1}$.

Proof. Denote by $D|_S$ the restriction of the divisor $D = \sum_{i=1}^{n} R_i$ to $S = \cup_{j=1}^{n+1}(P_j \cup Q_j)$. Then, applying Lemma 2.10 to $M = D - D|_S$, $N = D|_S + P_l - \sum_{j=1}^{n+1} P_j$, $L = L(z)$, and $\phi = \psi_i$, we get the assertion. □

Set

$$\phi_i(z, \lambda) = \frac{\rho_i(\nu_i)}{\psi_i(z, \nu_i)}\psi_i(z, \lambda).$$

Owing to (2.9), the corresponding pluriharmonic map $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ is given by

$$z = x + \sqrt{-1}y \mapsto [(\psi_{i,j}(z))],$$

where $\psi_{i,j}(z)$ is given by

$$\frac{\exp\left(\sum_{k=1}^{n+1}\left(\Lambda^{Q_k}(\eta_i)z_k\kappa_k^{-1} - \Lambda^{P_k}(\eta_i)\overline{z_k\kappa_k^{-1}}\right)\right)}{\exp\left(\sum_{k=1}^{n+1}\left(\Lambda^{Q_k}(\nu_i)z_k\kappa_k^{-1} - \Lambda^{P_k}(\nu_i)\overline{z_k\kappa_k^{-1}}\right)\right)} \cdot \frac{\prod_{k=0}^{i-1}\sigma^{\nu_k}(\eta_j)\cdot\prod_{k=i+1}^{n}\sigma^{\nu_k}(\eta_j)\prod_{k=1}^{n}\sigma^{R_k}(\nu_i)}{\prod_{k=1}^{n}\sigma^{R_k}(\eta_j)\prod_{k=1}^{i-1}\sigma^{\nu_k}(\nu_i)\prod_{k=i+1}^{n}\sigma^{\nu_k}(\nu_i)}. \tag{4.4}$$

Define a map $F\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by $(z_j)_j \mapsto (\kappa_j z_j)_j$ $(1 \leqq j \leqq k)$. Then the composition $\psi \circ F$ gives rise to the pluriharmonic map given in (3.12). This completes the proof of Theorem 3.11.

We now construct pluriharmonic maps corresponding to spectral data whose spectral curves are smooth elliptic curves, and prove Theorem 3.16.

Lemma 4.5. *Let* $\left(X = X_{\sqrt{-1}t}, \pi, \mathcal{O}_X(D) = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i)\right)$ *be a spectral data as in Theorem* 3.10*. Define a function* $\psi_i(z, u)$ *on* X *with parameter* z *by*

$$\psi_i(z,\, u) = \exp\left(\sum_{j=1}^{k} \frac{z_j}{\kappa_j}[\zeta_w(u - P_j) - A\,u] - \sum_{j=1}^{k} \overline{\left(\frac{z_j}{\kappa_j}\right)}[\zeta_w(u - Q_j) - A\,u]\right)$$

$$\cdot \frac{\prod_{j=1}^{d} \theta(F_j, u) \cdot \prod_{j=0}^{i-1} \theta(\nu_i, u) \cdot \theta(\widehat{\nu_i} + H(z), u) \prod_{j=i+1}^{n} \theta(\nu_i, u) \cdot}{\prod_{j=1}^{d+n+1} \theta(E_j, u)}. \tag{4.6}$$

Here ζ_w *is Weierstrass' zeta function as in* (3.7)*,*

$$H = H(z,\, \bar{z}) = \sum_{j=1}^{k} \frac{z_j}{\kappa_j} - \sum_{j=1}^{k} \overline{\left(\frac{z_j}{\kappa_j}\right)},$$

A *is the constant as in* (3.8)*, and* $\kappa_j = (\partial\,\zeta/\partial u)|_{u=P_j}$ *is the value of the differential of the meromorphic function* ζ *in* (2.7) *at* $u = P_j$*. Then* $\psi_l(z,\, u)\theta_A(z)$ *is an element of* $H^0(X,\, \mathcal{L}_i \otimes L(z))$ *for any* $z \in \mathbb{C}^k$*.*

Proof. Denote by $D|_S$ the restriction of the divisor $\sum_{i=1}^{d+n+1} E_i - \sum_{i=1}^{d} F_i$ to $S = \cup_{j=1}^{k}(P_j \cup Q_j)$. Then, applying Lemma 2.10 to $M = D - D|_S - \sum_{j=1}^{n+1-2k} P_{j+k}$, $N = D|_S + P_l - \sum_{j=1}^{k} 2P_j$, $L = L(z)$, and $\phi = \psi_i$, we get the assertion. □

Set $\phi_i(z, u) = \rho_i(\nu_i)\psi_i(z, u)/\psi_i(z, \nu_i)$. On account of (2.9), the corresponding pluriharmonic map $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ is given by

$$z = x + \sqrt{-1}y \mapsto [(\psi_{i,j}(z))],$$

where $\psi_{i,j}(z)$ is given by

$$\psi_{i,j}(z) = \frac{\mu_j}{\widetilde{\mu}_i} \frac{\exp\left(\sum_{k=1}^{n+1}\left([\zeta_w(\eta_j - P_k) - A\,\eta_j] z_k\kappa_k^{-1} - [\zeta_w(\eta_j - Q_k) - A\,\eta_j]\overline{z_k\kappa_k^{-1}}\right)\right)}{\exp\left(\sum_{k=1}^{n+1}\left([\zeta_w(\nu_i - P_k) - A\,\nu_i] z_k\kappa_k^{-1} - [\zeta_w(\nu_i - Q_k) - A\,\nu_i]\overline{z_k\kappa_k^{-1}}\right)\right)}$$

$$\cdot \frac{\prod_{k=0}^{n} \theta(R_k, \nu_i) \prod_{k=0}^{j-1} \theta(\nu_k, \eta_j)\theta(\widehat{\nu_i} + H(z), \eta_j) \prod_{k=j+1}^{n} \theta(\nu_k, \eta_j)}{\prod_{k=0}^{n} \theta(R_k, \eta_j) \prod_{k=0}^{j-1} \theta(\nu_k, \nu_i)\theta(\widehat{\nu_i} + H(z), \nu_i) \prod_{k=j+1}^{n} \theta(\nu_k, \nu_i).} \tag{4.7}$$

Define a map $F\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by $z_j \mapsto \kappa_j z_j$. Then the composition $\psi \circ F$ gives rise to the pluriharmonic map given in (3.17). This completes the proof of Theorem 3.16.

5. Computations of homomorphims into generalized Jacobians and periodicity conditions of pluriharmonic maps

McIntosh studied periodicity conditions of the corresponding pluriharmonic maps by introducing certain homomorphisms into generalized Jacobians. In this section, when X is a smooth rational or elliptic curve, we reformulate McIntosh's periodicity conditions by introducing certain families of hyperplanes on complex vector spaces, and prove Theorems 3.13 and 3.18.

Recall the following result:

Theorem 5.1 ([19]). *Pluriharmonic maps* $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ *corresponding to spectral data have a period* v *if* $\widehat{L}$ *has a period* v.

5.1. **The case of spectral data of type** $\mathrm{Pr_{n+1}}$ **with the rational spectral curve.** Let us determine the map $\widehat{L}$ when $(X,\, \pi,\, \mathcal{L})$ is a spectral data with the smooth rational curve as its spectral curve.

First, we compute the map $L\colon \mathbb{C}^{n+1} \to J(X)$ defined by $z = x + \sqrt{-1}y \mapsto L(z)$. In the case of the smooth rational curve X, $L(z)$ is always the trivial line bundle over X.

Then

$$\exp\left(\sum_{j=1}^{k}\left(\Lambda^{Q_j}(\lambda)\frac{z_j}{\kappa_j} - \Lambda^{P_j}(\lambda)\overline{\left(\frac{z_j}{\kappa_j}\right)}\right)\right)\otimes\theta_A(z) \tag{5.2}$$

belongs to $H^0(X, \mathcal{O}_X)$ by Lemma 2.10. Moreover we see that this is a non-vanishing global holomorphic section of $\mathcal{O}_X$.

Using (5.2) and $H^0(X,\, \mathcal{O}_X) \cong \mathbb{C}$, we see that

$$\theta_A(z) = C\exp\left(\sum_{j=1}^{n+1}\left(-\Lambda^{Q_j}(\lambda)\frac{z_j}{\kappa_j} + \Lambda^{P_j}(\lambda)\overline{\left(\frac{z_j}{\kappa_j}\right)}\right)\right)$$

where C is a non-zero constant.

Now we give an explicit description of $\widehat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{o}})_{\mathbb{R}} = J(X_{\mathfrak{n}})_{\mathbb{R}} \times J(X_{\mathfrak{m}})_{\mathbb{R}}$.

For $0 \leqq i \leqq n$, we define $A_i \in \mathrm{Hom}\,(\mathcal{O}|_{\nu_i},\, \mathcal{O}_X|_{\nu_0})$ by the condition that each A_i maps the element $1|_{\nu_i}$ of $\mathcal{O}_X|_{\nu_i}$ to the element $1|_{\nu_0}$ of $\mathcal{O}_X(-T)|_{\nu_0}$.

Replacing ν_i with η_i for $0 \leqq i \leqq n$, we get $B_i \in \mathrm{Hom}\,(\mathcal{O}_X|_{\eta_i},\, \mathcal{O}_X|_{\eta_0})$. Then the map $\widehat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{n}})_{\mathbb{R}} \times J(X_{\mathfrak{m}})_{\mathbb{R}}$ is given by

$$z = x + \sqrt{-1}y \mapsto \Big((h_i(z,\, \bar{z}))_{1\leqq i\leqq n}, (g_i(z,\, \bar{z}))_{1\leqq i\leqq n}\Big), \tag{5.3}$$

where $g_i(z,\, \bar{z})$ and $h_i(z,\, \bar{z})$ are elements of $\mathrm{Hom}\,(\mathcal{O}|_{\nu_i},\, \mathcal{O}_X|_{\nu_0})$ and $\mathrm{Hom}\,(\mathcal{O}_X|_{\eta_i},\, \mathcal{O}_X|_{\eta_0})$ being defined by

$$g_i(z,\, \bar{z}) = \exp(a_i(z,\bar{z}))\, A_i, \quad h_i(z,\, \bar{z}) = \exp(b_i(z,\bar{z}))\, B_i,$$

respectively. Here a_i is defined by

$$a_i(z,\, \bar{z}) = \sum_{j=1}^{n+1}\left(-\frac{z_j}{\kappa_j}[\Lambda^{Q_j}(\nu_0) - \Lambda^{Q_j}(\nu_i)] + \overline{\left(\frac{z_j}{\kappa_j}\right)}[\Lambda^{P_j}(\nu_0) - \Lambda^{P_j}(\nu_i)]\right).$$

and b_i is defined by replacing ν_i with η_i in this equation.

Lemma 5.4. *For $1 \leqq i \leqq n$, $a_i(z, \bar{z})$ and $b_i(z, \bar{z})$ are pure imaginary.*

Proof. Using $\overline{\nu_i} = 1/\nu_i$ and $\overline{\eta_i} = 1/\eta_i$, we get the desired result. □

Thus we can consider $\widehat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{n}})_{\mathbb{R}} \times J(X_{\mathfrak{m}})_{\mathbb{R}}$ to be a map $L_T\colon \mathbb{C}^{n+1} \to T^{2n} = S^1 \times \cdots \times S^1$ defined by

$$z \mapsto (\exp(a_1(z, \bar{z})), \ldots , \exp(a_n(z, \bar{z}), \exp(b_1(z, \bar{z})), \ldots , \exp(b_n(z, \bar{z}))) .$$

We can also consider $pr\colon J(X_{\mathfrak{o}})_{\mathbb{R}} \to \mathbb{P}U_{n+1}$ to be a map $pr\colon J(X_{\mathfrak{o}})_{\mathbb{R}} \to \mathbb{P}U_{n+1}$. Then pr is given as follows

$$\left((\alpha_p A_p)_{p=1,\ldots,n}, (\beta_p B_p)_{p=1,\ldots,n}\right) \mapsto \left[\left(\frac{\beta_j}{\alpha_i} P_{i,j}\right)\right]_{0\leqq i,j\leqq n}, \tag{5.5}$$

where $\alpha_0 = \beta_0 = 1$ and $P_{i,j}$ is given by

$$\frac{\prod_{k=0}^{i-1} \sigma^{\nu_k}(\eta_j) \cdot \prod_{k=i+1}^{n} \sigma^{\nu_k}(\eta_j) \prod_{k=1}^{n} \sigma^{R_k}(\nu_i)}{\prod_{k=1}^{n} \sigma^{R_k}(\eta_j) \prod_{k=1}^{i-1} \sigma^{\nu_k}(\nu_i) \prod_{k=i+1}^{n} \sigma^{\nu_k}(\nu_i)}. \tag{5.6}$$

Proposition 5.7. *The composition $pr \circ \hat{L}\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ is same as in* (4.7)*.*

Proof. Substituting $\exp(a_i)$ and $\exp(b_i)$ into α_i and β_i respectively, we get the desired result. □

Evidently, $\widehat{L}$ is doubly periodic if and only if L_T is doubly periodic. Then we have the following

Proposition 5.8. *The pluriharmonic map $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ defined by* (4.7)*, corresponding to a spectral data $(X, \pi, \mathcal{L})$ is a lift of a map from $\mathbb{C}^k/\Gamma$ with a p-dimensinal lattice $\Gamma = \oplus_{l=1}^{p} \subset \mathbb{C}^{n+1}$ if the set $V = \bigcap_{1\leqq i\leqq 2n} V_i$ contains Γ, where $V_1, \ldots , V_{2n}$ are the sets defined by*

$$V_i = \{z \in \mathbb{C}^k \mid f_i(z) = \sum_{l=1}^{k} \gamma_{i,l} z_l \in \pi\left(\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}\right)\} \tag{5.9}$$

where $f_0, \ldots f_{2n}$ are linear holomorphic functions on $\mathbb{C}^{n+1}$ defined by

$$\gamma_{i,l} = \begin{cases} \Lambda^{Q_l}(\nu_i)/\kappa_l & (1 \leqq i \leqq n) \\ \Lambda^{Q_l}(\eta_i)/\kappa_l & (n+1 \leqq i \leqq 2n). \end{cases}$$

Proof. Recall that ψ has a lattice Γ as periods if L_T has a lattice Γ as periods by Theorem 5.1. If L_T has Γ as periods, then Γ is contained in V, since V is the set of all points on which the value of L_T is equal to the initial value $L_T(0) = (1, \ldots , 1) \in T^n$.

Conversely, if V contains a lattice Γ, then clearly $v_1, \dots, v_p$ are periods of L_T, since L_T is a homomorphism from the additive group $\mathbb{C}^k$ to T^n. Hence Condition (**??**) is a necessary and sufficient condition for L_T to be periodic with periods $v_1, \dots, v_p$.

□

Now let us prove Theorem 3.13.

Proof of Theorem 3.13. From the argument in the proof of Theorem 3.11, we see that the map given in Theorem 3.13 is a composition $\psi \circ F$, where ψ is the map as in (4.4) and F is a map defined by $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, $(z_1, \dots, z_{n+1}) \mapsto (\kappa_1 z_1, \dots, \kappa_{n+1} z_{n+1})$. Thus Theorem 3.13 follows immediately from Proposition **??**. □

5.2. **The case of spectral data of type** $\mathrm{Pr}_{\mathrm{n+1}}$ **with elliptic spectral curves.** Let us determine the map $\widehat{L}$ when $(X, \pi, \mathcal{L})$ is a spectral data with a smooth elliptic curve as its spectral curve. First, we compute the map $L\colon \mathbb{C}^{n+1} \to J(X)$ defined by $z = x + \sqrt{-1}y \mapsto L(z)$. Let T_z be a divisor defined by

$$T_z = (0) - (H(z, \bar{z})), \tag{5.10}$$

where H is a point on X givne in Lemma 4.5. Let Θ be a meromorphic function on $\mathbb{C}^2$ defined by

$$\Theta(w, u) = \frac{\theta(0, u)}{\theta(w, u)}.$$

Then

(5.11)

$$\exp\left(\sum_{j=1}^{n+1} \frac{z_j}{\kappa_j}[\zeta_w(u - P_j) - A\,u] - \sum_{j=1}^{n+1} \overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u - Q_j) - A\,u]\right) \Theta(H(z, \bar{z}), u)^{-1} \otimes \theta_A(z)$$

belongs to $H^0(X, \mathcal{O}_X(-T_z) \otimes L(z))$ by Lemma 2.10. Moreover we see that this is a non-vanishing global holomorphic section of $\mathcal{O}_X(T_z) \otimes L(z)$. In particular, the line bundle $L(z) \otimes \mathcal{O}_X(T_z)$ is trivial, that is, $L(z) \otimes \mathcal{O}_X(T_z) \cong \mathcal{O}_X$, and hence $L(z) \cong \mathcal{O}_X(-T_z)$. Using (5.10) and identifying Jacobian $J(X)$ with $X \cong \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}t\mathbb{Z})$, we see that $L\colon \mathbb{C}^{n+1} \to J(X)$ is given by

$$z = x + \sqrt{-1}y \mapsto H(z, \bar{z}) - 0 = H(z,\, \bar{z}) = \sum_{j=1}^{n+1} (z_j/\kappa_j - \overline{(z_j/\kappa_j)}) \mod \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}t,$$

where κ_j is the complex number in Lemma 4.5.

Using (5.11) and $H^0(X,\, L(z)\otimes \mathcal{O}_X(T_z)) = H^0(X,\, \mathcal{O}_X) \cong \mathbb{C}$, we see that

$$\theta_A(z) = C\exp\left(-\sum_{j=1}^{n+1}\frac{z_j}{\kappa_j}[\zeta_w(u-P_j) - A\,u] + \sum_{j=1}^{n+1}\overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u-Q_j) - A\,u]\right)\Theta_1\left(H(z,\bar z), u\right),$$

where C is a non-zero constant.

Now we give an explicit description of $\widehat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{o}})_{\mathbb{R}} = J(X_{\mathfrak{n}})_{\mathbb{R}} \times J(X_{\mathfrak{m}})_{\mathbb{R}}$. Let $v\colon S^1_J = \{e^{\sqrt{-1}\theta} \mid \theta\in\mathbb{R}\} \to J(X)_{\mathbb{R}}$ be a map defined by

$$e^{\sqrt{-1}\theta} \mapsto S(\theta) = \sqrt{-1}t\theta/2\pi \mod \mathbb{Z}\oplus\mathbb{Z}\sqrt{-1}t.$$

Let $J^n_S \to S^1_J$ and $J^m_S \to S^1_J$ be the pull-backs of $J(X_{\mathfrak{n}})_{\mathbb{R}}$ and $J(X_{\mathfrak{m}})_{\mathbb{R}}$ by v, respectively. For $0 \leqq i \leqq n$, we define $A_i\colon e^{\sqrt{-1}\theta} \in S^1_J \mapsto A_i(e^{\sqrt{-1}\theta}) \in \mathrm{Hom}\left(v(e^{\sqrt{-1}\theta})|_{\nu_i},\, v(e^{\sqrt{-1}\theta})|_{\nu_0}\right)$, sections of $J^n_S \to S^1_J$, by the condition that each $A_i(e^{\sqrt{-1}\theta})$ maps the element

$$\exp(\sqrt{-1}\nu_i\theta)\Theta(\sqrt{-1}t\theta/(2\pi), \nu_i)$$

of $\mathcal{O}_X(-T_z)|_{\nu_i}$ to the element

$$\exp(\sqrt{-1}\nu_0\theta)\Theta(\sqrt{-1}t\theta/(2\pi), \nu_0)$$

of $\mathcal{O}_X(-T_z)|_{\nu_0}$. Replacing ν_i with η_i for $0 \leqq i \leqq n$, we get $B_i\colon e^{\sqrt{-1}\theta} \in S^1_J \mapsto B_i(e^{\sqrt{-1}\theta}) \in \mathrm{Hom}\left(v(e^{\sqrt{-1}\theta})|_{\eta_i},\, v(e^{\sqrt{-1}\theta})|_{\eta_0}\right)$, sections of $J^m_S \to S^1_J$. Since the image of $\mathbb{C}^{n+1}$ by L is contained in $\mathbb{Z}\oplus\mathbb{R}\tau \mod \mathbb{Z}\oplus\mathbb{Z}\tau$ ($\subset J(X)$), we can regard $\widehat{L}\colon \mathbb{C}^{n+1} \to J(X_{\mathfrak{o}})_{\mathbb{R}}$ as a map $\mathbb{C}^{n+1} \to J^n_S \times J^m_S$. Using this identification, the map $\widehat{L}\colon \mathbb{C}^{n+1} \to J_S$ is given by

$$z = x + \sqrt{-1}y \mapsto \left((h_i(z,\,\bar z))_{1\leqq i\leqq n}, (g_i(z,\,\bar z))_{1\leqq i\leqq n}\right), \tag{5.12}$$

where $g_i(z,\,\bar z)$ and $h_i(z,\,\bar z)$ are elements of $\mathrm{Hom}\,(v(\exp(2\pi H(z,\,\bar z)/t))|_{\nu_i},\, v(\exp(2\pi H(z,\,\bar z)/t))|_{\nu_0})$ and $\mathrm{Hom}\,(v(\exp(2\pi H(z,\,\bar z)/t))|_{\eta_i},\, v(\exp(2\pi H(z,\,\bar z)/t))|_{\eta_0})$ being defined by

$$g_i(z,\,\bar z) = \exp(a_i(z,\bar z))\, A_i(\exp(2\pi H(z,\,\bar z)/t)),$$
$$h_i(z,\,\bar z) = \exp(b_i(z,\bar z))\, B_i(\exp(2\pi H(z,\,\bar z)/t)),$$

respectively. Here a_i is defined by

$$\begin{aligned} a_i(z,\,\bar z) = &-\sum_{j=1}^{n+1}\frac{z_j}{\kappa_j}[\zeta_w(\nu_0 - P_j) - \zeta_w(\nu_i - P_j) - \frac{B}{\tau}\,(\nu_0-\nu_i)] \\ &+\sum_{j=1}^{n+1}\overline{(\frac{z_j}{\kappa_j})}[\zeta_w(\nu_0 - Q_j) - \zeta_w(\nu_i - Q_j) - \frac{B}{\tau}\,(\nu_0-\nu_i)] \end{aligned}$$

and b_i is defined by replacing ν_i with η_i in this equation.

Lemma 5.13. *For $1 \leqq i \leqq n$, $a_i(z,\bar z)$ and $b_i(z,\bar z)$ are pure imaginary.*

Proof. We may assume that $0 \leqq \operatorname{Im} P_0, \operatorname{Im} Q_0, \operatorname{Im}\eta_0, \ldots, \operatorname{Im}\eta_n < \operatorname{Im}\tau$. On this assumption, $Q_0 = \overline{P_0} + \tau$. Using $\overline{\zeta_w(u)} = \zeta_w(\bar{u})$ and $\bar{B} = -B$, we then get

$$\begin{aligned}
&\overline{[\zeta_w(\eta_0 - P_j) - \zeta_w(\eta_i - P_j) - B\tau^{-1}(\eta_0 - \eta_i)]} \\
&\quad = [\zeta_w(\overline{\eta_0 - P_j}) - \zeta_w(\overline{\eta_i - P_j})] - B\tau^{-1}\overline{(\eta_0 - \eta_i)} \\
&\quad = [\zeta_w(\overline{\eta_0} - Q_j + \tau) - \zeta_w(\overline{\eta_i} - Q_j + \tau)] - B\tau^{-1}\overline{(\eta_0 - \eta_i)}.
\end{aligned} \tag{5.14}$$

In the case that $\eta_0 \in S^1_A$ and $\eta_i \in S^1_B$, it follows from $\zeta_w(u+\tau) = \zeta_w(u) + B$ that the right hand side of (5.14) is equal to

$$\begin{aligned}
&[\zeta_w(\eta_0 - Q_j + \tau) - \zeta_w(\eta_i - \tau - Q_j + \tau)] - B\tau^{-1}(\eta_0 - \eta_i + \tau) \\
&\quad = [\zeta_w(\eta_0 - Q_j) - \zeta_w(\eta_i - Q_j)] - B\tau^{-1}(\eta_0 - \eta_i),
\end{aligned}$$

which implies see that b_i is pure imaginary. Similarly, we can also see that b_i is pure imaginary in other cases. ☐

Thus we can consider $\widehat{L}\colon \mathbb{C}^{n+1} \to J^n_S \times J^m_S$ to be a map $L_T\colon \mathbb{C}^{n+1} \to T^{2n+1} = S^1_J \times S^1 \times \cdots \times S^1$ defined by

$$z \mapsto (\exp(2\pi H(z,\bar{z})/t), \exp(a_1(z,\bar{z})), \ldots, \exp(a_n(z,\bar{z})), \exp(b_1(z,\bar{z})), \ldots, \exp(b_n(z,\bar{z}))).$$

We can also consider $pr\colon J(X_{\mathfrak{o}})_{\mathbb{R}} \to \mathbb{P}U_{n+1}$ to be a map $pr\colon J_S \to \mathbb{P}U_{n+1}$. Then pr is given as follows

$$\left(\left(\alpha_p A_p(\exp(\sqrt{-1}\theta))\right)_{p=1,\ldots,n}, \left(\beta_p B_p(\exp(\sqrt{-1}\theta))\right)_{p=1,\ldots,n}\right) \mapsto \left[\left(\frac{\beta_j}{\alpha_i} P_{i,j}\right)\right]_{0\leqq i,j\leqq n}, \tag{5.15}$$

where $\alpha_0 = \beta_0 = 1$ and $P_{i,j}$ is given by

$$\begin{aligned}
P_{i,j}(z) = {} & \frac{\mu_j}{\widetilde{\mu_i}} \frac{\exp(\sqrt{-1}(\eta_0 - \eta_j)\theta)}{\exp(\sqrt{-1}(\nu_0 - \nu_i)\theta)} \\
& \cdot \frac{\prod_{k=0}^n \theta(R_k, \nu_i) \prod_{k=0}^{j-1} \theta(\nu_k, \eta_j)\theta(\widehat{\nu_i} + S(\theta), \eta_j) \prod_{k=j+1}^n \theta(\nu_k, \eta_j)}{\prod_{k=0}^n \theta(R_k, \eta_j) \prod_{k=0}^{j-1} \theta(\nu_k, \nu_i)\theta(\widehat{\nu_i} + S(\theta), \nu_i) \prod_{k=j+1}^n \theta(\nu_k, \nu_i).}
\end{aligned} \tag{5.16}$$

Proposition 5.17. *The composion $pr \circ \hat{L}\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ is same as in* (4.7).

Proof. Substituting $\exp(a_i)$, $\exp(b_i)$ and $2\pi H(z,\bar{z})/(\sqrt{-1}t)$ into α_i, β_i and θ respectively, we get the desired result. ☐

Evidently, $\widehat{L}$ is doubly periodic if and only if L_T is doubly periodic. Then we have the following

Proposition 5.18. *The pluriharmonic map* $\psi\colon \mathbb{C}^{n+1} \to \mathbb{P}U_{n+1}$ *defined by* (4.7)*, corresponding to a spectral data* $(X, \pi, \mathcal{L})$ *is a lift of a map from* $\mathbb{C}^{n+1}/\Gamma$ *with a p-dimensinal lattice* $\Gamma = \oplus_{l=1}^{p}\mathbb{Z}v_l \subset \mathbb{C}^{n+1}$ *if the set* $V = \bigcap_{0\leqq i\leqq 2n} V_i$ *contains* Γ*, where* $V_0, \dots , V_{2n}$ *are the sets defined by*

$$V_i = \{z \in \mathbb{C}^k \mid f_i(z) = \sum_{l=1}^{k} \gamma_{i,l} z_l \in \pi \left(\mathbb{R} \oplus \sqrt{-1}\mathbb{Z}\right)\} \tag{5.19}$$

where $f_0, \dots f_{2n}$ *are linear holomorphic functions on* $\mathbb{C}^{n+1}$ *defined by*

$$\gamma_{i,l} = \begin{cases} ([\zeta_w(\nu_0 - P_l) - \zeta_w(\nu_i - P_l) - B\,(\nu_0 - \nu_i)\tau^{-1}])/\kappa_l & (1 \leqq i \leqq n) \\ ([\zeta_w(\eta_0 - P_l) - \zeta_w(\eta_i - P_l) - B\,(\eta_0 - \eta_i)\tau^{-1}])/\kappa_l & (n+1 \leqq i \leqq 2n) \\ 2\pi/(\kappa_l t) & (i = 0). \end{cases}$$

Proof. Recall that ψ has a lattice Γ as periods if L_T has a lattice Γ as periods by Theorem 5.1. If L_T has Γ as periods, then Γ is contained in V, since V is the set of all points on which the value of L_T is equal to the initial value $L_T(0) = (1, \dots , 1) \in T^{2n+1}$.

Conversely, if V contains a lattice Γ, then clearly $v_1, \dots , v_p$ are periods of L_T, since L_T is a homomorphism from the additive group $\mathbb{C}^{n+1}$ to T^{2n+1}. Hence Condition (5.19) is a necessary and sufficient condition for L_T to be doubly periodic with periods $v_1, \dots , v_p$. □

Now let us prove Theorem 3.18.

Proof of Theorem 3.18. From the argument in the proof of Theorem 3.16, we see that the map given in Theorem 3.18 is a composition $\psi \circ F$, where ψ is the map in Proposition 5.17 and F is a map defined by $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, $(z_j)_j \mapsto (\kappa_j z_j)_j$. Thus Theorem 3.18 follows immediately from Proposition 5.18. □

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