An exposition of McIntosh's work by showing many examples

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§1 Introduction

I. McIntosh [Mc1] gave a 1-1 correspondence between the spectral data $\{\pi : X \to \mathbf{P}^1, \mathcal{L}\}$ which satisfies the certain conditions and the linearly full non-isotropic harmonic maps of finite type $\mathbf{R}^2 \to \mathbf{C}P^n$. If $\varphi : \mathbf{R}^2 \to \mathbf{C}P^n$ is a harmonic map of isotropy order r ($1 \leqq r < \infty$) then there is a certain harmonic map $\psi : \mathbf{R}^2 \to F^r(\mathbf{C}P^n)$ into a flag manifold such that $\varphi = p \circ \psi$, where $p : F^r(\mathbf{C}P^n) \longrightarrow \mathbf{C}P^n$ is the homogeneous projection.

Fact (1) $r = 1$ if and only if φ is non-conformal. In this case, we have $F^1(\mathbf{C}P^n) = \mathbf{C}P^n$.

(2) $r = \infty$ if and only if φ is isotropic.

Now, we represent $F^r(\mathbf{C}P^n) = G/K$ as a homogeneous space and denote by $\mathcal{G}$ and $\mathcal{K}$ the Lie algebras of G and K, respectively. Then we have the canonical decomposition $\mathcal{G} = \mathcal{K} + \mathcal{M}$. It is known that $F^r(\mathbf{C}P^n)$ have the structure of $(r+1)$-symmetric space in the sense of O. Kowalski[K]. Let τ be the automorphism of order $(r+1)$ with fixed set $\mathcal{K}$ on G/K which gives the $(r+1)$-symmetric space structure on G/K. Let $\mathcal{G}_i$ be the ω^i-eigenspace of τ, where $\omega = \exp(2\pi\sqrt{-1}/(r+1))$. Then we have

$$\begin{cases} \mathcal{G}^{\mathbf{C}} = \sum_{i=0}^{r} \mathcal{G}_i, \quad \mathcal{K}^{\mathbf{C}} = \mathcal{G}_0, \quad \mathcal{M}^{\mathbf{C}} = \sum_{i=1}^{r} \mathcal{G}_i, \\ \mathcal{G}_{-i} = \overline{\mathcal{G}_i}, \quad [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}, \qquad \text{(index is regarded as } \textit{mod } k) \end{cases}$$

The map $\mathcal{G} \longrightarrow T_x(G/K)$ given by $\xi \mapsto \frac{d}{dt}\,|_{t=0} \exp t\xi \cdot x$ restricts to an isomorphism $\mathrm{Ad}g \cdot \mathcal{M} \longrightarrow T_x(G/K)$. We denote the inverse map by $\beta : T_x(G/K) \longrightarrow \mathrm{Ad}g \cdot \mathcal{M} \subset \mathcal{G}$ and we may regard β as a $\mathcal{G}$-valued 1-form on G/K, which is called *Maurer-Cartan form* for G/K. Denote by $[\mathcal{G}_i]$ the vector bundle over G/K of which the fibre at $x = g \cdot o \in G/K$ is given by $\mathrm{Ad}g \cdot \mathcal{G}_i$.

Definition. ψ is said to be *primitive* if $(\psi^*\beta)(\frac{\partial}{\partial z})$ is $[\mathcal{G}_1]$- valued, where β is the Maurer-Cartan form for G/K.

Take a lift $F : \mathbf{R}^2 \to G$ of ψ with $\psi = \tilde{\pi} \circ F$, where $\tilde{\pi} : G \to G/K$ the natural projection. Then we see that $\psi^*\beta = \mathrm{Ad}F \cdot \alpha_{\mathcal{M}}$, where $\alpha = F^{-1}dF$ and $\alpha = \alpha_{\mathcal{K}} + \alpha_{\mathcal{M}}$ is a decomposition of α with respect to the canonical decomposition $\mathcal{G} = \mathcal{K} + \mathcal{M}$. Therefore, we see that ψ is primitive if and only if $\alpha_{\mathcal{M}}(\frac{\partial}{\partial z})$ is $\mathcal{G}_1$-valued.

Remark. Black[B] proved that if $r \geq 2$ then a primitive map ψ is a harmonic map with respect to any invariant metric on G/K. In case of $r = 1$, the condition of the primitivity is always satisfied and meaningless. Hence, we suppose that $\psi = \varphi$ is a harmonic map when $r = 1$. When we treat both cases in a unified way, we call such ψ a *primitive harmonic map*. We denote by $\alpha_{\mathcal{M}} = \alpha'_{\mathcal{M}} + \alpha''_{\mathcal{M}}$ the type decomposition of $\alpha_{\mathcal{M}}$ according to the decomposition of the complexified cotangent bundle of the domain manifold. Note that $\alpha''_{\mathcal{M}}(\partial/\partial\bar{z}) = -\big(\alpha'_{\mathcal{M}}(\partial/\partial z)\big)^*$.

Example 1.1. Define $\alpha'_{\mathcal{M}}(\partial/\partial z)$ and $\alpha'_{\mathcal{K}}(\partial/\partial z)$ with $K = U(1) \times U(1) \times U(1)$ by

$$\alpha'_{\mathcal{M}}(\partial/\partial z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha'_{\mathcal{K}}(\partial/\partial z) = \mathbf{0}.$$

Set $F = \exp(z\alpha'_{\mathcal{M}}(\partial/\partial z)) \cdot \exp(\bar{z}\alpha''_{\mathcal{M}}(\partial/\partial\bar{z}))$. Then $\psi = \tilde{\pi} \circ F : \mathbf{R}^2 \longrightarrow F^2(\mathbf{C}P^2) = SU(3)/T^2$ is a primitive harmonic map into a full flag manifold and $\varphi = p \circ \psi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^2$ is a harmonic map of isotropy order 2. The ψ is a primitive map corresponding to the vacuum solution(cf. [BP]). Note that φ is indeed doubly-periodic.

§2 **Spectral curves**

Let X and Y be compact Riemann surfaces of genus g and genus g', respectively. Let $\pi : X \longrightarrow Y$ be a holomorphic covering map. Denote by R the ramification divisor of π. Then we have a *Riemann-Hurwitz formula* :

$$2g - 2 = \deg(\pi) \cdot (2g' - 2) + \deg(R).$$

Now, take $Y = \mathbf{C}P^1$, hence $g' = 0$. Consider $\pi : X \longrightarrow \mathbf{C}P^1$ defined by $\zeta \mapsto \pi(\zeta) = \zeta^{n+1} = \lambda$. Then the degree of π is $(n+1)$ and the branch points of π are $\zeta = 0$ and $\zeta = \infty$ with ramification indices $n+1$. In this case, the ramification divisor R is

$$R = n(0) + n(\infty),$$

hence $\deg(R) = 2n$. Therefore, the Riemann-Hurwitz formula tells us that the genus of X is zero, i.e., a Riemann sphere $\mathbf{C}P^1$.

The interpretation of the primitive harmonic map equation for $\psi : M \longrightarrow G/K$ by the pullback of the Maurer-Cartan form is as follows :

$$\text{(2.1)} \qquad \begin{cases} d\alpha'_{\mathcal{M}} + [\alpha''_{\mathcal{K}} \wedge \alpha'_{\mathcal{M}}] = 0, \qquad \alpha'_{\mathcal{M}} \text{ is } \mathcal{G}_1 - \text{valued}, \\ d\alpha'_{\mathcal{K}} + [\alpha'_{\mathcal{K}} \wedge \alpha''_{\mathcal{K}}] + [\alpha'_{\mathcal{M}} \wedge \alpha''_{\mathcal{M}}] = 0. \end{cases}$$

In the following examples, we will show how one obtains a spectral curve X from a primitive harmonic map of finite type.

Example 2.1. Define $\alpha'_{\mathcal{M}}(\partial/\partial z)$ and $\alpha'_{\mathcal{K}}(\partial/\partial z)$ with $K = U(1)\times U(1)\times U(1)$ by

$$\alpha'_{\mathcal{M}}(\partial/\partial z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha'_{\mathcal{K}}(\partial/\partial z) = \mathbf{0}.$$

Then, as in Example 1.1, we have a harmonic map $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^2$ with framing $F = \exp(z\alpha'_{\mathcal{M}}(\partial/\partial z)) \cdot \exp(\overline{z}\alpha''_{\mathcal{M}}(\partial/\partial \overline{z}))$. Set $\alpha'_\eta = \eta\alpha'_{\mathcal{M}} + \alpha'_{\mathcal{K}} = \eta\alpha'_{\mathcal{M}}$ and define α_η by $\alpha_\eta = \alpha'_\eta - (\alpha'_\eta)^*$. Moreover, define F_η and ξ by

$$F_\eta = \exp(z\alpha'_\eta(\partial/\partial z) - \overline{z}\big(\alpha'_\eta(\partial/\partial z)\big)^*), \quad \xi = \mathrm{Ad}F_\eta^{-1}\cdot \alpha_\eta.$$

Then we see that the following equation holds :

$$d\xi = [\xi, F_\eta^{-1}dF_\eta].$$

In this situation, ξ is called a *polynomial Killing field* and hence $\psi : \mathbf{R}^2 \longrightarrow F^2(\mathbf{C}P^2)$ is a primitive harmonic map of *finite type*. Note that any harmonic map φ with isotropy order r of 2-torus into $\mathbf{C}P^n$ is obtained as $\varphi = p\circ\psi$ for some primitive harmonic map ψ of finite type into $F^r(\mathbf{C}P^n)$. Now, the spectral curve X is defined by the equation $\det(\alpha'_\eta - \zeta I) = 0$. Setting $\lambda = \eta^3$, a simple calculation shows that $\lambda = \zeta^3$. Then, the Riemann-Hurwitz formula yields that $X = \mathbf{C}P^1$ a rational curve.

Example 2.2. Define $\alpha'_{\mathcal{M}}(\partial/\partial z)$ and $\alpha'_{\mathcal{K}}(\partial/\partial z)$ with $K = U(1)\times U(1)\times U(2)$ by

$$\alpha'_{\mathcal{M}}(\partial/\partial z) = \begin{pmatrix} 0 & 0 & b & -\overline{a} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha'_{\mathcal{K}}(\partial/\partial z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & \overline{b} \end{pmatrix},$$

where $\mid a\mid^2 + \mid b\mid^2 = 1$ and $0 <\mid b \mid < 1$. Set $\alpha' = \alpha'_{\mathcal{M}} + \alpha'_{\mathcal{K}}$ and $F = \exp(z\alpha'(\partial/\partial z)\cdot \exp(-\overline{z}\big(\alpha'(\partial/\partial z)\big)^*)$. Then $\psi = \tilde{\pi}\circ F : \mathbf{R}^2 \longrightarrow F^2(\mathbf{C}P^3) = U(4)/U(1)\times U(1)\times U(2)$ is a primitive harmonic map and $\varphi = p\circ\psi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^3$ is a weakly conformal harmonic map of isotropy order 2. Set $\alpha'_\eta = \alpha'_{\mathcal{K}} + \eta\alpha'_{\mathcal{M}}$ and $\alpha_\eta = \alpha'_\eta - \big(\alpha'_\eta\big)^*$. As in Example 2.1, we see that ψ is of finite type. The equation $\det(\alpha'_\eta(\partial/\partial z) - \zeta I) = 0$ with $\lambda = \eta^3$ yields

$$\lambda = \frac{1}{b}\frac{\zeta^3(\zeta - \overline{b})}{(\zeta - b^{-1})}.$$

The branch points are $\zeta = 0, \zeta = \infty$ with ramification index 3 and the others are $\zeta = \alpha_\pm$, where

$$\alpha_\pm = \frac{\mid b\mid^2 + 2 \pm \sqrt{\mid b\mid^4 - 5\mid b\mid^2 + 4}}{3b}.$$

Therefore, the ramification divisor R is given by

$$R = 2(0) + (\alpha_+) + (\alpha_-) + 2(\infty).$$

Hence, $\deg(R) = 6$ and the Riemann-Hurwitz formula yields that the genus of the spectral curve X is zero because the degree of π is four. Moreover, the *real structure* ρ_X of X is defined by $\rho_X(\zeta, \lambda) = (\overline{\zeta}^{-1}, \overline{\lambda}^{-1})$, which induces a real structure $\rho : \lambda \mapsto \overline{\lambda}^{-1}$ on $\mathbf{C}P^1$. Notice that the preimage of the equator S^1 of $\mathbf{C}P^1$ is ρ_X-fixed and there is no branch points on the preimage of the equator.

Remark. T.Taniguchi[T1] determined all the smooth spectral curves of genus zero.

Example 2.3. Define α'_η by

$$\alpha'_\eta = \eta \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} + \eta^2 \begin{pmatrix} \sqrt{-1/2} & 0 \\ 0 & -\sqrt{-1/2} \end{pmatrix} + \eta^3 \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix},$$

where $K = U(1) \times U(1)$. Set $\alpha_\eta = \alpha'_\eta - \left(\alpha'_\eta\right)^*$. Then, we have a decomposition $\exp(\overline{z}\eta^{-2}\alpha_\eta) = F_\eta \cdot G_\eta$, where $F_\eta : \mathbf{R}^2 \longrightarrow \Lambda G_\tau$, $G_\eta : \mathbf{R}^2 \longrightarrow \Lambda G_\tau^{\mathbf{C}}$(see [BP]). If we set $\xi = \mathrm{Ad}F_\eta^{-1} \cdot \alpha_\eta$ then we have $d\xi = [\xi, F_\eta^{-1}dF_\eta]$, hence ξ is a polynomial Killing field and $\varphi = \psi = \tilde{\pi} \circ F_\eta \mid_{\eta=1}: \mathbf{R}^2 \longrightarrow \mathbf{C}P^1$ is a harmonic map of finite type. Now, let me calculate the spectral curve. The equation $\det(\alpha'_\eta - \zeta I) = 0$ yields that $\zeta^2 = \lambda(\lambda-2)(\lambda-1/2)$ with $\lambda = \eta^2$. Thus, the spectral curve X is an elliptic curve, i.e., of genus one. In fact, the ramification divisor R of π is given by

$$R = (\lambda = 0) + (\lambda = 2) + (\lambda = \frac{1}{2}) + (\lambda = \infty),$$

hence $\deg(R) = 4$ and the Riemann-Hurwitz formula yields that the genus of X is one because the degree of π is two. The real structure ρ_X of X is given by $\rho_X(\zeta, \lambda) = (\overline{\zeta\lambda}^{-2}, \overline{\lambda}^{-1})$. Notice that the preimage of the equator of $\mathbf{C}P^1$ is ρ_X-fixed and there is no branch points on the preimage of the equator.

In general, if one finds a polynomial Killing field for the primitive harmonic map of finite type then one can define the spectral curve from the characteristic polynomial of it(see [H]). However, it is not so easy to deduce a *spectral data* from given primitive harmonic map of finite type. This is the reason why the section 7 of McIntosh's paper[Mc1] is lengthy.

§3 **Spectral data**

First of all, we give a definition of the spectral data :

Definition A triple $(X, \pi, \mathcal{L})$ is said to be a *spectral data* if they satisfy the following conditions :

(1) X is a complete connected algebraic curve of arithmetic genus p with real structure

ρ_X, i.e., anti-holomorphic involution,
(2) there is a holomorphic covering map $\pi : X \longrightarrow \mathbf{C}P^1$ with $\deg(\pi) = n+1$ such that $\pi \circ \rho_X = \overline{\pi}^{-1}$(=a real structure of $\mathbf{C}P^1$) and the divisor of π is given by

$$(\pi) = (m+1)P_0 + P_1 + \cdots + P_{n-m} - (m+1)Q_0 - Q_1 - \cdots - Q_{n-m},$$

where $Q_j = \rho_X(P_j)$ for $j = 0, 1, \cdots, n-m$,
(3) $\mathcal{L}$ is a (complex) line bundle over X of degree $(n+p)$ such that

$$f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \longrightarrow \mathcal{O}_X(R)$$

is a ρ_X-equivariant($\overline{\rho_{X*}f} = f$) isomorphism, where R is the ramification divisor of π and $\mathcal{O}_X(R)$ is the line bundle corresponding to R,
(4) ρ_X fixes each point of the preimage $X_{\mathbf{R}}$ of the equator S^1_λ of $\mathbf{C}P^1_\lambda$, π has no branch points on $X_{\mathbf{R}}$ and f is non-negative on $X_{\mathbf{R}}$.

Given a spectral data $(X, \pi, \mathcal{L})$, we see from the Riemann-Hurwitz formula that $\deg(R) = 2n + 2p$. Then, the condition (4) above guarantees that there is some positive divisor D on X with $\deg(D) = n+p$ such that $R = D + \rho_{X*}(D)$. If we identify $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D_0)$ for some divisor D_0 on X, the condition (3) above implies that f is a rational function on X with a divisor (f) given by $(f) = (D_0 + \rho_{X*}(D_0) - D - \rho_{X*}(D))$.

Now, let me introduce a Hermitian inner product on $H^0(X, \mathcal{L})$, which is the vector space of all global holomorphic sections of $\mathcal{L}$. Denote by $\pi_*\mathcal{L}$ the direct image sheaf of $\mathcal{L}$ by π, i.e., $\Gamma(U, \pi_*\mathcal{L}) = \Gamma(\pi^{-1}(U), \mathcal{L})$ for any open subset U of $\mathbf{C}P^1_\lambda$. Then it follows from Grothendieck's Riemann-Roch theorem that $H^0(X, \mathcal{L}) = H^0(\mathbf{C}P^1_\lambda, \pi_*\mathcal{L})$. Set $A = \mathbf{C}P^1_\lambda \setminus \{0, \infty\}$ and $I = I_0 \cup I_\infty$, where I_0 (resp. I_∞) is an open neighborhood around 0 (resp. around ∞) which contains no branch points except 0 (resp. except ∞). Hence, $\mathbf{C}P^1_\lambda = A \cup I$. Next, set $X_A = \pi^{-1}(A)$ and $X_I = \pi^{-1}(I)$ so that $X = X_A \cup X_I$.

Definition. Define a bilinear form h on $\Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L})$ by

$$h : \Gamma(X_A, \mathcal{L}) \times \Gamma(X_A, \mathcal{L}) \ni (v, w) \mapsto \mathrm{Tr}(f \cdot v \otimes \overline{\rho_{X*}w}) \in \mathbf{C}[\lambda^{-1}, \lambda],$$

where $\mathbf{C}[\lambda^{-1}, \lambda]$ is the ring generated by λ, λ^{-1} over the field $\mathbf{C}$. Notice that $f \cdot v \otimes \overline{\rho_{X*}w}$ is a holomorphic section of $\mathcal{O}_X(R)$ over X_A, whence its trace is a holomorphic function on A. Take a point P of $\mathbf{C}P^1_\lambda$. Then, we have

$$h(v, w)(P) = \sum_{x \in \pi^{-1}(P)} f(x) \cdot v(x)\overline{w(\rho_X(x))}.$$

The summation is taken over all points $\{x_0, \cdots, x_n\} = \pi^{-1}(P)$ and it is counted with multiplicities if P is a (or an image of) branch point. The obvious properties of h is as

follows : For any holomorphic function b on A we have
(1) $h(bv,w)=bh(v,w)$, (2) $h(v,bw)=\overline{\rho_{\mathbf{C}P^1_\lambda *}b}h(v,w)$, (3) $h(w,v)=\overline{\rho_{\mathbf{C}P^1_\lambda *}h(v,w)}$.
Now, we may regard $H^0(X,\mathcal{L})$ as a subset of $\Gamma(X_A,\mathcal{L})$. Since a global holomorphic function on $\mathbf{C}P^1_\lambda$ is a constant and f is ρ_X- equivariant, i.e., $\overline{\rho_{X*}f}=f$, we have $\overline{\rho_{\mathbf{C}P^1_\lambda *}b}=\overline{b}$ and $\rho_{\mathbf{C}P^1_\lambda *}h(v,w)=h(v,w)$, hence we see that $h\mid_{H^0\times H^0}$ defines a Hermitian symmetric form. The positive definiteness of $h\mid_{H^0\times H^0}$ depends on the choice of f. Since $h\mid_{H^0\times H^0}$ is a constant, it is enough to evaluate it at some point of $\mathbf{C}P^1$. In particular, evaluating it at the equator S^1_λ we see from the non-negativity of f on $X_{\mathbf{R}}$(condition (4) of the spectral data) that $h\mid_{H^0\times H^0}$ is positive definite, hence it defines a Hermitian inner product.

Now, let $\{\eta_0,\cdots,\eta_n\}$ be the inverse image of $1\in S^1_\lambda$. They are $(n+1)$-distinct ρ_X-fixed points due to the condition (4) of the spectral data. Take a local frame $S_0,\cdots,S_n$ such that $S_i(\eta_j)=0$ for $i\neq j$ and $f(\eta_i)\cdot S_i(\eta_i)\otimes\overline{\rho_{X*}(S_i(\eta_i))}=1$ for $i=0,\cdots,n$. If we represent v,w around $\pi^{-1}(1)$ as

$$v=\sum_{i=0}^n v^iS_i,\quad w=\sum_{i=0}^n w^iS_i,$$

then we have

$$\begin{aligned}h(v,w)&=\sum_{x\in\pi^{-1}(1)} f(x)\cdot v(x)\otimes\overline{w(\rho_X(x))}\\&=\sum_{j=0}^n f(\eta_j)\cdot\big(\sum_{i=0}^n v^iS_i(\eta_j)\big)\overline{\sum_{k=0}^n w^kS_k(\rho_X(\eta_j))}\\&=\sum_{j=0}^n v^j\overline{w^j}.\end{aligned}$$

We state a lemma on the triviality of $\pi_*\mathcal{L}$, of which the proof we omit since it takes us to a rather lengthy trip :

Lemma 3.1. *$\pi_*\mathcal{L}$ is a rank $(n+1)$ trivial bundle over $\mathbf{C}P^1_\lambda$.*

Since $h^0(X,\mathcal{L})=\dim H^0(X,\mathcal{L})=\dim H^0(\mathbf{C}P^1_\lambda,\pi_*\mathcal{L})=n+1$, it follows from the Riemann-Roch formula

$$h^0(X,\mathcal{L})-h^1(X,\mathcal{L})=1-p+\deg(\mathcal{L})=n+1$$

that $h^1(X,\mathcal{L})=0$, in which case $\mathcal{L}$ is called *non-special.*

We give some examples of the spectral data.

Example 3.1. Consider $\pi(\zeta)=\zeta^4=\lambda$. Then, $X=\mathbf{C}P^1_\zeta$ and $\pi:\mathbf{C}P^1_\zeta\longrightarrow\mathbf{C}P^1_\lambda$ is a degree 4 holomorphic covering map. The divisor of π is

$$(\pi)=4(0)-4(\infty).$$

The ramification divisor R of π is given by

$$R = 3(0) + 3(\infty).$$

The real structure ρ_X is given by $\rho_X(\zeta, \lambda) = (\overline{\zeta}^{-1}, \overline{\lambda}^{-1})$, which induces a real structure $\rho_{\mathbf{C}P^1_\lambda}$ on $\mathbf{C}P^1_\lambda$. Now, take a line bundle $\mathcal{L}$ as $\mathcal{L} = \mathcal{O}_X(3(0))$. Then, we have $\deg(\mathcal{L}) = 3$ and obviously $\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} = \mathcal{O}_X(R)$, whence we take $f = 1$. Then, the conditons (3) and (4) of the spectral data are satisfied. The vector space $H^0(X, \mathcal{L})$ is generated by $\{1, \zeta^{-1}, \zeta^{-2}, \zeta^{-3}\}$ over X_A and $h^0(X, \mathcal{L}) = 4$. Of course, if we denote by $\{x_0, x_1\}$ the homogeneous coordinate for $\mathbf{C}^2$ then

$$H^0(X, \mathcal{L}) = \mathrm{Span}\{x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3\}.$$

If $x_0 \neq 0$ then, setting $\zeta = x_0/x_1$, we have the generating basis above. If we retake $f = 1/4$ then the generating basis above are orthonormal basis with respect to $h\,|_{H^0 \times H^0}$.

Example 3.2. Consider $\pi(\zeta) = \frac{1}{\alpha}\zeta^3 \frac{(\zeta-\alpha)}{(\zeta-\alpha^{-1})} = \lambda$, where $0 < \alpha < 1$(cf. Example 2.2). Then, as in Example 2.2, we see that $X = \mathbf{C}P^1_\zeta$ and $\pi : X \longrightarrow \mathbf{C}P^1_\lambda$ is a degree 4 holomorphic covering map. The real structure ρ_X is given by $\rho_X(\zeta, \lambda) = (\overline{\zeta}^{-1}, \overline{\lambda}^{-1})$. The divisor of π is

$$(\pi) = 3(0) + (\alpha) - (\alpha^{-1}) - 3(\infty).$$

The ramification divisor R of π is given by

$$R = 2(0) + (p) + (p^{-1}) + 2(\infty),$$

where

$$p = \frac{\alpha^2 + 2 - \sqrt{\alpha^4 - 5\alpha^2 + 4}}{3\alpha}.$$

Therefore, there is no branch points on $X_{\mathbf{R}}$. Now, take a line bundle $\mathcal{L}$ as $\mathcal{L} = \mathcal{O}_X(3(\infty))$ and take f as

$$f(\zeta) = \frac{-\zeta}{(\zeta - p)(\zeta - p^{-1})}.$$

Then $f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \longrightarrow \mathcal{O}_X(R)$ is a ρ_X-equivariant isomorphism. We see that $(f) = (0) + (\infty) - (p) - (p^{-1})$ and f is non-negative on $X_{\mathbf{R}}$. Let $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the elements of $\pi^{-1}(1)$. Then, we easily see that $\zeta_i = \overline{\zeta_i}^{-1}$ for $i = 1, \cdots, 4$. More generally, each point of $X_{\mathbf{R}}$ is ρ_X-fixed. In fact, solve the equation $\zeta^4 - \alpha\zeta^3 - \alpha e^{\sqrt{-1}\theta}\zeta + e^{\sqrt{-1}\theta} = 0$. Notice that this equation is invariant under the tansformation $\zeta \mapsto \overline{\zeta}^{-1}$. We define a local frame S_1, S_2, S_3, S_4 over X_A as follows :

$$S_1 = \sqrt{f(\zeta_1)}\frac{(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}{(\zeta_1 - \zeta_2)(\zeta_1 - \zeta_3)(\zeta_1 - \zeta_4)}, \quad S_2 = \sqrt{f(\zeta_2)}\frac{(\zeta - \zeta_1)(\zeta - \zeta_3)(\zeta - \zeta_4)}{(\zeta_2 - \zeta_1)(\zeta_2 - \zeta_3)(\zeta_2 - \zeta_4)},$$
$$S_3 = \sqrt{f(\zeta_3)}\frac{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_4)}{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)(\zeta_3 - \zeta_4)}, \quad S_4 = \sqrt{f(\zeta_4)}\frac{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)}{(\zeta_4 - \zeta_1)(\zeta_4 - \zeta_2)(\zeta_4 - \zeta_3)}.$$

These satisfy $S_i(\zeta_j) = 0$ for $i \neq j$ and $f(\zeta_i)S_i(\zeta_i)\overline{\rho_{X*}S_i(\zeta_i)} = 1$ for $i = 1, \cdots, 4$.

§4 **A parallel transport**

To construct a map from $\mathbf{R}^2$ and to define a conncetion for a line bundle over $\mathbf{R}^2$, we need to define a parallel transport of a section of $\mathcal{L}$ to a section of a line bundle over $\mathbf{R}^2$. Let $J(X)$ be the Jacobian variety of the spectral curve X, i.e., $J(X) = H^1(X, \mathcal{O})/H^1(X, \mathbf{Z})$, which is a p-dimensional complex torus, where p is the genus of X, and defined by the long exact sequence coming from the short exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0.$$

The set of all line bundles $L \in J(X)$ which satisfy $\overline{\rho_{X*}L} \cong L^{-1}$ forms a subgroup of $J(X)$ by a tensor product. We denote by $J_{\mathbf{R}}(X)$ the connected component of the identity of this subgroup(the identity is trivial line bundle). Then, $J_{\mathbf{R}}(X)$ is a p-dimensional real torus. For any $L \in J_{\mathbf{R}}(X)$, we see that a line bundle $\mathcal{L}\otimes L$ satisfies $(\mathcal{L}\otimes L)\otimes\overline{\rho_{X*}(\mathcal{L}\otimes L)} \cong \mathcal{O}_X(R)$. In this case, we say that $\mathcal{L}\otimes L$ is *real*. Note that when we replace $\mathcal{L}$ by $\mathcal{L}\otimes L$ for $L \in J_{\mathbf{R}}(X)$ we see that f is still non-negative on the preimage $X_{\mathbf{R}}$ of the equator S^1_λ. In fact, f is independent of L. Since $\deg(\mathcal{L}\otimes L) = \deg(\mathcal{L}) = n + p$, it follows from Lemma 3.1 that $\pi_*(\mathcal{L}\otimes L)$ is a rank $(n+1)$ trivial bundle and $h^0(X, \mathcal{L}\otimes L) = n + 1$. Now, consider a complex vector bundle $H^0(X) \mapsto J_{\mathbf{R}}(X)$ of which the fibre at $L \in J_{\mathbf{R}}(X)$ is given by a $(n+1)$-dimensional complex vector space $H^0(X, \mathcal{L}\otimes L)$. Recall that $X = X_A \cup X_I$. A line bundle $L \in J(X)$ is trivialized over X_A or X_I. We denote by θ_A and θ_I trivializing sections over X_A and X_I, respectively, i.e.,

$$L\mid_{X_A} \stackrel{\theta_A}{\cong} X_A \times \mathbf{C}, \quad L\mid_{X_I} \stackrel{\theta_I}{\cong} X_I \times \mathbf{C}.$$

Over $X_A \cap X_I$, we have a transition relation $\theta_I = e^a\theta_A$. Thus, for $L \in J_{\mathbf{R}}(X)$, we have a 1-cocycle (e^a, X_A, X_I). Conversely, a 1-cocycle (e^a, X_A, X_I) defines a line bundle L with e^a as a transition function. Then, consider a map $L : \mathcal{G} = \Gamma(X_A \cap X_I, \mathcal{O}_X) \longrightarrow J(X)$ defined by $a \mapsto L(a)$, where $L(a)$ is a line bundle with a transition function e^a. Set

$$\mathcal{G}_{\mathbf{R}} = \{a \in \mathcal{G} \mid \overline{\rho_{X*}a} = -a\}.$$

Then, we see that $\mathrm{Im}(L\mid_{\mathcal{G}_{\mathbf{R}}}) = J_{\mathbf{R}}(X)$. Now, fix a trivializing section θ for $\mathcal{L}$ over X_I such that $\mathrm{Tr}(f\cdot\theta\otimes\overline{\rho_{X*}\theta}) = 1$. For $a \in \mathcal{G}_{\mathbf{R}}$, set $\theta_a = \theta\otimes\theta_I$, which is a trivializing section for $\mathcal{L}\otimes L(a)$ over X_I. We want to define a map

$$\iota_a : \Gamma(X_A, \mathcal{L}\otimes L(a)) \longrightarrow \Gamma(X_A, \mathcal{L}).$$

Lemma 4.1. *For $\sigma_a \in \Gamma(X_A, \mathcal{L} \otimes L(a))$, define $\iota_a(\sigma_a)$ by*

$$\iota_a(\sigma_a) = e^a(\sigma_a \theta_a^{-1})\theta.$$

Then, we have $\iota_a(\sigma_a) \in \Gamma(X_A, \mathcal{L})$.

Proof. Let τ be a trivializing section of $\mathcal{L}$ over X_A. We may write $\theta = e^c\tau$. Therefore, we have $\theta_a = e^{a+c}\tau \otimes \theta_A$. Now, we calculate

$$\begin{aligned} \iota_a(\sigma_a) &= e^a(\sigma_a \theta_a^{-1})\theta \\ &= e^{-c}\sigma_a(\tau \otimes \theta_A)^{-1}\theta = \sigma_a(\tau \otimes \theta_A)^{-1}\tau, \end{aligned} \tag{4.1}$$

where $\sigma_a(\tau \otimes \theta_a)^{-1}$ is a holomorphic function over X_A and τ is a trivializing section of $\mathcal{L}$ over X_A, thus we have $\iota_a(\sigma_a) \in \Gamma(X_A, \mathcal{L})$. q.e.d.

Indeed, $\iota_a : \Gamma(X_A, \mathcal{L} \otimes L(a)) \longrightarrow \Gamma(X_A, \mathcal{L})$ is an isomorphism. The injectivity of ι_a is obvious. To show the surjectivity of ι_a, take an arbitrary $\sigma \in \Gamma(X_A, \mathcal{L})$. Then, we may write $\sigma = b\tau$ for some $b \in \Gamma(X_A, \mathcal{O})$. We choose $\sigma_a = b(\tau \otimes \theta_A)$. Then we have $\iota_a(\sigma_a) = b\tau = \sigma$ by (4.1), proving the surjectivity of ι_a. Let $L^*H^0(X) \mapsto \mathcal{G}_{\mathbf{R}}$ be the pullback bundle of the bundle $H^0(X) \mapsto J_{\mathbf{R}}(X)$ by $L : \mathcal{G}_{\mathbf{R}} \longrightarrow J_{\mathbf{R}}(X)$. Let $\{\tau_0, \cdots, \tau_n\}$ be an orthonormal frame of global sections of $\mathcal{L}$. We have $\Gamma(X_A, \mathcal{L}) = \mathrm{Span}\{\tau_0, \cdots, \tau_n\} \mid_{X_A}$, because $\Gamma(X_A, \mathcal{L})$ is a free $\mathcal{O}$-module of rank $(n+1)$ by the facts that $\Gamma(X_A, \mathcal{L}) = \Gamma(A, \pi_*\mathcal{L})$ and $\pi_*\mathcal{L}$ is a trivial bundle of rank $(n+1)$. Any element of $\Gamma(X_A, \mathcal{L})$ is expressed as $\sum \sigma_j(\lambda)\tau_j$. Define an evaluation map $ev_1 : \Gamma(X_A, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L})$ by $\sum \sigma_j(\lambda)\tau_j \mapsto \sum \sigma_j(1)\tau_j$, where $\sigma_j(1)$ is the value of $\sigma_j(\lambda)$ at $\lambda = 1$ and a constant. Now, the composition $ev_1 \circ \iota_a \mid_{H^0(X, \mathcal{L} \otimes L(a))}$: $H^0(X, \mathcal{L} \otimes L(a)) \longrightarrow H^0(X, \mathcal{L})$ is an isomorphism. Indeed, it is clearly surjective by the way of construction of the map and it is injective by the fact that $h^0(X, \mathcal{L} \otimes L(a)) = h^0(X, \mathcal{L}) = n + 1$.

Lemma 4.2. *Let $\sigma_1, \sigma_2 \in H^0(X, \mathcal{L} \otimes L(a))$ and set $s_j = \iota_a(\sigma_j)$ for $j = 1, 2$. Then $h(s_1, s_2)$ is a constant.*

Proof. For simplicity, set $\mathcal{L}(a) = \mathcal{L} \otimes L(a)$. The map $\iota_a : \Gamma(X_A, \mathcal{L}(a)) \longrightarrow \Gamma(X_A, \mathcal{L})$ induces an isomorphism $\square_a : \Gamma(X_A, \mathcal{L}(a) \otimes \overline{\rho_{X*}\mathcal{L}(a)}) \longrightarrow \Gamma(X_A, \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}})$. In fact, $\square_a(\sigma) = \sigma(\theta_a \otimes \overline{\rho_{X*}\theta_a})^{-1}\theta \otimes \overline{\rho_{X*}\theta}$ because the transition function e^a for $L(a)$ and the transition function e^{-a} for $\overline{\rho_{X*}L(a)}$ cancel out each other. Set $s_{12} = \square_a(\sigma_1 \otimes \overline{\rho_{X*}\sigma_2})$, which is a section of $\Gamma(X_A, \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}})$. We claim that s_{12} is a globally defined holomorphic section of $\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}}$. First of all, s_{12} is holomorphic over X_A. Next, to show that it is also holomorphic over X_I, set $f_j = \sigma_j\theta_a^{-1}$ for $j = 1, 2$, which is a holomorphic function on X_I. Now, we have

$$\begin{aligned} s_{12} &= \sigma_1 \otimes \overline{\rho_{X*}\sigma_2}(\theta_a \otimes \overline{\rho_{X*}\theta_a})^{-1}\theta \otimes \overline{\rho_{X*}\theta} \\ &= f_1\theta \otimes \overline{\rho_{X*}(f_2\theta)}. \end{aligned}$$

Therefore, we see that s_{12} is also holomorphic over X_I since θ is a holomorphic frame field over X_I. Thus, s_{12} is a globally holomorphic section, proving the claim. Then we have

$$\begin{aligned} h(s_1, s_2) &= h(\iota_a(\sigma_1), \iota_a(\sigma_2)) \\ &= \mathrm{Tr}(f \cdot \iota_a(\sigma_1) \otimes \overline{\rho_{X*}\iota_a(\sigma_2)}) \\ &= \mathrm{Tr}(f \cdot \Box_a(\sigma_1 \otimes \overline{\rho_{X*}\sigma_2})) = \mathrm{Tr}(f \cdot s_{12}). \end{aligned}$$

Since $\mathrm{Tr}(f \cdot s_{12}) \in H^0(\mathbf{C}P^1_\lambda, \mathcal{O})$ (notice that $f \cdot s_{12} \in H^0(X, \mathcal{O}_X(R))$), we see that $h(s_1, s_2)$ is a constant. q.e.d.

§5 The construction of harmonic maps into $\mathbf{C}P^n$

Consider a map $a : \mathbf{R}^2 \longrightarrow \mathcal{G}_{\mathbf{R}}$ defined by $z \mapsto a(z, \overline{z}) = z\zeta^{-1} - \overline{z}\zeta$, where ζ is considered only on $X_A \cap (U_0 \cup U_\infty)$, where U_0 (resp. U_∞) is a connected component of X_0 (resp. X_∞) which contains P_0 (resp. Q_0). Then $L(a) = L(z\zeta^{-1} - \overline{z}\zeta)$ is a 2-parameter subgroup of $J_{\mathbf{R}}(X)$. We have a diagram

$$\begin{array}{ccccc} L(a)^* H^0(X) & \longrightarrow & L^* H^0(X) & \longrightarrow & H^0(X) \supset H^0(X, \mathcal{L} \otimes L) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}^2 & \xrightarrow{a} & \mathcal{G}_{\mathbf{R}} & \xrightarrow{L} & J_{\mathbf{R}}(X) \ni L \end{array}$$

We also rewrite $L(a)^* H^0(X)$ as $H^0(X)$ if there is no confusion. Fix h-orthonormal basis $\{\tau_j\}$ for $H^0(X, \mathcal{L})$ such that $(H^0(X, \mathcal{L}), h) \longrightarrow (\mathbf{C}^{n+1}, <\ ,\ >)$ is isometric. We want to decompose the vector bundle $H^0(X) \mapsto \mathbf{R}^2$ into line subbundles which are orthogonal to each other. For the purpose, first define the following line bundles for which the sheaves of germs of holomorphic sections are subsheaves of the sheaf of germs of holomorphic sections for $\mathcal{L}$:

$$(5.1) \quad \begin{cases} \mathcal{L}_j = \mathcal{L} \otimes \mathcal{O}_X(-(m-j)P_0 - jQ_0 - \sum_{i=1}^{n-m} P_i) & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \mathcal{L}_m = \mathcal{L} \otimes \mathcal{O}_X(-mQ_0). \end{cases}$$

Lemma 5.1. *For $j = 0, 1, \cdots, m$, each $\mathcal{L}_j$ is non-special, i.e., $h^1(X, \mathcal{L}_j) = 0$.*

Proof. Set $\mathcal{I}_j = \mathcal{L} \otimes \mathcal{O}_X(jP_0 - jQ_0)$ for $j = 0, 1, \cdots, m$. Then, this is a real line bundle, i.e., it satisfies $\mathcal{I}_j \otimes \overline{\rho_{X*}\mathcal{I}_j} \cong \mathcal{O}_X(R)$. It follows from Lemma 3.1 that $\pi_*\mathcal{I}_j$ is a rank $(n+1)$ trivial bundle. Define $\mathcal{F}_j$ by $\mathcal{F}_j = \mathcal{I}_j \otimes \mathcal{O}_X(-(m+1)P_0 - \sum_{i=1}^{n-m} P_i)$. Then we obtain

$$\mathcal{F}_j = \begin{cases} \mathcal{L}_j(-P_0) & \text{for} \quad j = 0, \cdots, m-1, \\ \mathcal{L}_m(-P_0 - \sum_{i=1}^{n-m} P_i) & \text{for} \quad j = m. \end{cases}$$

Note that $\deg(\mathcal{F}_j) = p-1$ for $j = 0, 1, \cdots, m$. In general, we know that $H^0(X, L(-P)) \cong \{s \in H^0(X, L) \mid s(P) = 0\}$, where $L(D) = L \otimes \mathcal{O}_X(D)$ for a divisor D on X. In fact, if we fix a meromorphic section τ with the divisor $(\tau) = (-P)$ then tensoring each element by τ or τ^{-1} gives an isomorphism. Now, suppose that $\mathcal{F}_j$ has a non-trivial global section. Then there is a global section of $\pi_*\mathcal{I}_j$ which vanishes at $\lambda = 0$ because any global holomorphic section of $\mathcal{F}_j$ is a global holomorphic section of $\mathcal{I}_j$ with divisor $(m+1)P_0 + \sum P_i$. However, since $\pi_*\mathcal{I}_j$ is a trivial bundle, it must be identically zero. Thus, we see that $h^0(X, \mathcal{F}_j) = 0$. Now, the Riemann-Roch formula implies that $h^1(X, \mathcal{F}_j) = 0$ because $\deg(\mathcal{F}_j) = p-1$ for $j = 0, 1, \cdots, m$. In general, for any line bundle L and any point $P \in X$, $h^1(X, L) = 0$ implies that $h^1(X, L(P)) = 0$. Indeed, it follows from the Serre duality theorem that $0 = h^1(X, \mathcal{L}) = h^0(X, \Omega_X^{1,0} \otimes L^{-1})$, where $\Omega_X^{1,0}$ is the holomorphic cotangent bundle (=canonical bundle) of X. Again, from the Serre duality it follows that $h^1(X, L(P)) = h^0(X, \Omega_X^{1,0} \otimes L^{-1}(-P))$. Therefore, if there is a non-trivial element of $H^1(X, L(P))$ then there is a global section of $\Omega_X^{1,0} \otimes L^{-1}$ which vanishes at P. However, it must be identically zero because $h^0(X, \Omega_X^{1,0} \otimes L^{-1}) = 0$. For the completion of the proof, notice that $\mathcal{L}_j = \mathcal{F}_j(P_0)$ for $j = 0, 1, \cdots, m-1$ and $\mathcal{L}_m = \mathcal{F}_m(P_0 + \sum P_i)$. Apply the general theory above to these line bundles once or successively. q.e.d.

Corollary 5.1. *For arbitrary $a \in \mathcal{G}_{\mathbf{R}}$, we have $h^1(X, \mathcal{L}_j \otimes L(a)) = 0$ for $j = 0, 1, \cdots, m$.*

Proof. Just replace $\mathcal{L}$ by $\mathcal{L} \otimes L(a)$ in the definition of $\mathcal{I}_j$ in the proof of Lemma 5.1. q.e.d.

Corollary 5.1, together with the Riemann-Roch theorem, yields that

$$h^0(X, \mathcal{L}_j \otimes L(a)) = \begin{cases} 1 & \text{for} \quad j = 0, 1, \cdots, m-1, \\ n+1-m & \text{for} \quad j = m. \end{cases}$$

Then, we obviously obtain

$$H^0(X, \mathcal{L} \otimes L(a)) = \bigoplus_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a)) \qquad \text{(direct sum)}.$$

Define a map $\tau^1 : \Gamma(X_A, \mathcal{L}) \longrightarrow \mathbf{C}^{n+1}$ by the composition of the identification $H^0(X, \mathcal{L})$ with $\mathbf{C}^{n+1}$ by $\{\tau_j\}$ and the map ev_1. We draw a diagram of our present situation :

$$\begin{array}{l} \Gamma(X_A, \mathcal{L}) \xrightarrow{ev_1} H^0(X, \mathcal{L}) \xrightarrow{\{\tau_j\}} \mathbf{C}^{n+1} \\ \quad \iota_a \Big\uparrow \\ \Gamma(X_A, \mathcal{L} \otimes L(a)) \supset H^0(X, \mathcal{L} \otimes L(a)) = \bigoplus_{j=0}^{m} H^0(X, \mathcal{L}_j \otimes L(a)) \end{array}$$

Define a line subbundle l_j of the trivial bundle $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ by

$$l_j = \tau^1 \circ \iota_a(H^0(X, \mathcal{L}_j \otimes L(a)) \qquad \text{for} \quad j = 0, 1, \cdots, m.$$

Then we claim that $\mathbf{R}^2 \times \mathbf{C}^{n+1} = \bigoplus_{j=0}^m l_j$, which is an orthogonal direct sum with respect to the inner product $< , >$ on $\mathbf{C}^{n+1}$. To show the claim, it is enough to prove the following lemma :

Lemma 5.2. *For $j = 0, 1, \cdots, m$, let $\sigma_j \in H^0(X, \mathcal{L}_j \otimes L(a))$ with $z, \bar{z}$ fixed. Set $s_j = \iota_a(\sigma_j)$. Then $h(s_j, s_k) = 0$ for $j \neq k$.*

Proof. From Lemma 4.2 we know that $h(s_j, s_k)$ is a constant. Therefore, it is enough to show that when $j \neq k$, $h(s_j, s_k)$ is zero at some point of $\mathbf{C}P^1_\lambda$. If we set $f_j = \sigma_j \theta_a^{-1}$ then we see that f_j is a holomorphic function over X_I and $s_j = e^a(\sigma_j \theta_a^{-1})\theta = e^a f_j \theta$. Since σ_j is a global holomorphic section of $\mathcal{L} \otimes L(a)$ which has a divisor $(m-j)P_0 + jQ_0 + \sum P_i$ for $j = 0, \cdots, m-1$ or a divisor mQ_0 for $j = m$, it follows that f_j has a divisor

$$\begin{cases} (m-j)P_0 + jQ_0 + \sum P_i & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \quad mQ_0 & \text{for} \quad j = m. \end{cases}$$

Set $r_{jk} = f \cdot \theta \otimes \overline{\rho_{X*}\theta} f_j \overline{\rho_{X*} f_k}$. Then we have $h(s_j, s_k) = \mathrm{Tr}(r_{jk})$. Recall that $X_I = X_0 \cup X_\infty$. We denoted by U_0 (resp. U_∞) a connected component of X_0 (resp.X_∞) which contains P_0 (resp. Q_0). Remember that there is no branch points on X_0 and X_∞ except P_0 and Q_0. It follows from the fact that $f \cdot \theta \otimes \overline{\rho_{X*}\theta}$ is a meromorphic function with a divisor $(-R)$ that

$$f \cdot \theta \otimes \overline{\rho_{X*}\theta} = \begin{cases} \zeta^{-m} & \text{in} \quad U_0, \\ \zeta^m & \text{in} \quad U_\infty, \\ 1 & \text{elsewhere} \quad \text{in} \quad X_I. \end{cases}$$

Therefore, r_{jk} has a divisor

$$(5.2) \quad \begin{cases} (k-j)P_0 + (j-k)Q_0 + \sum P_i + \sum Q_i & \text{for} \quad j, k = 0, 1, \cdots, m-1, \\ (m-j)P_0 + (j-m)Q_0 + \sum P_i & \text{for} \quad k = m; j = 0, 1, \cdots, m-1. \end{cases}$$

Note that $\pi^{-1}(0) = \{(m+1)P_0, P_1, \cdots, P_{n-m}\}$ and $\pi^{-1}(\infty) = \{(m+1)Q_0, Q_1, \cdots, Q_{n-m}\}$, where $(m+1)P_0$ (resp. $(m+1)Q_0$) is a point P_0 (resp. Q_0) with multiplicity $(m+1)$. This, together with (5.2), yields that if $j, k < m$

$$\mathrm{Tr}(r_{jk}) = \sum_{\pi^{-1}(0)} r_{jk} = 0 \qquad \text{when} \quad k > j,$$

$$\mathrm{Tr}(r_{jk}) = \sum_{\pi^{-1}(\infty)} r_{jk} = 0 \qquad \text{when} \quad j > k,$$

if $j < k = m$

$$\mathrm{Tr}(r_{jm}) = \sum_{\pi^{-1}(0)} r_{jm} = 0,$$

proving our assertion. q.e.d.

Lemma 5.3. *Let $\sigma : \mathbf{R}^2 \longrightarrow H^0(X)$ be a smooth section for which $\sigma(z, \overline{z})$ is a globally holomorphic section of $\mathcal{F} \otimes L(a)$ for some ideal sheaf $\mathcal{F}$ of $\mathcal{L}$. Let D be the covariant derivation on $H^0(X)$ induced by the parallel transport. Then, $D_{\partial/\partial z}\sigma$ (resp. $D_{\partial/\partial \overline{z}}\sigma$) is a globally defined holomorphic section of $\mathcal{F}(P_0) \otimes L(a)$ (resp. $\mathcal{F}(Q_0) \otimes L(a)$).*

Remark. Each $\mathcal{L}_j$ is an ideal sheaf of $\mathcal{L}$.

Proof. We can define a connection D on the bundle $H^0(X)$ by

$$D_Z\sigma = \iota_a^{-1}(Z\iota_a(\sigma)),$$

where σ is a section of the bundle $H^0(X) \mapsto \mathbf{R}^2$ and Z is an arbitrary vector field on $\mathbf{R}^2$. If we set $s = \iota_a(\sigma)$ and $f = \sigma\theta_a^{-1}$ then we see that $s = e^a f\theta : \mathbf{R}^2 \longrightarrow \Gamma(X_A, \mathcal{F})$ and $f : \mathbf{R}^2 \longrightarrow \Gamma(X_I, \mathcal{F} \otimes \mathcal{L}^{-1})$. Recall that $a = z\zeta^{-1} - \overline{z}\zeta$. We obtain

$$\begin{cases} \dfrac{\partial s}{\partial z} = \zeta^{-1}e^a f\theta + \dfrac{\partial f}{\partial z}e^a\theta = (\zeta^{-1}f + \dfrac{\partial f}{\partial z})e^a\theta \in \Gamma(X_A, \mathcal{F}(P_0)), \\ \dfrac{\partial s}{\partial \overline{z}} = -\zeta e^a f\theta + \dfrac{\partial f}{\partial \overline{z}}e^a\theta = (-\zeta f + \dfrac{\partial f}{\partial \overline{z}})e^a\theta \in \Gamma(X_A, \mathcal{F}(Q_0)). \end{cases}$$

Thus, from the definition of D it follows that

$$\begin{cases} D_{\partial/\partial z}\sigma = (\zeta^{-1}f + \dfrac{\partial f}{\partial z})\theta_a \in \Gamma(X_I, \mathcal{F}(P_0) \otimes L(a)), \\ D_{\partial/\partial \overline{z}}\sigma = (-\zeta f + \dfrac{\partial f}{\partial \overline{z}})\theta_a \in \Gamma(X_I, \mathcal{F}(Q_0) \otimes L(a)). \end{cases}$$

Since $\partial s/\partial z$ is holomorphic over X_A, $D_{\partial/\partial z}\sigma$ is also holomorphic over X_A. Therefore, $D_{\partial/\partial z}\sigma$ is holomorphic over $X = X_A \cup X_I$ and a globally defined holomorphic section of $\mathcal{F}(P_0) \otimes L(a)$. The case of $D_{\partial/\partial \overline{z}}\sigma$ is similar. q.e.d.

Let $\sigma_0, \sigma_1, \cdots, \sigma_n$ be a global section of $H^0(X) \mapsto \mathbf{R}^2$ for which $H^0(X, \mathcal{L}_j \otimes L(a)) = \mathrm{Span}\{\sigma_j\}$ for $j = 0, 1, \cdots, m-1$ and $H^0(X, \mathcal{L}_m \otimes L(a)) = \mathrm{Span}\{\sigma_m, \cdots, \sigma_n\}$. Set $s_j = \iota_a(\sigma_j)$ for $j = 0, 1, \cdots, n$. Then, $s_0, \cdots, s_n$ is a free system of generators for $\Gamma(X_A, \mathcal{L})$. Let $B = \mathbf{C}[\lambda, \lambda^{-1}]$ be the ring generated by λ, λ^{-1}. Let V_j and V_m be B-modules generated, respectively, by s_j and $s_m, \cdots, s_n$, where $j = 0, 1, \cdots, m-1$. We have

$$\Gamma(X_A, \mathcal{L}) = \sum_{j=0}^{m} V_j,$$

which is a h-orthogonal direct sum by Lemma 5.2. We denote by $\Pi_j : \Gamma(X_A, \mathcal{L}) \longrightarrow V_j$ a h-orthogonal projection onto V_j.

Lemma 5.4. *The map* $s_j : \mathbf{R}^2 \longrightarrow \Gamma(X_A, \mathcal{L})$ *satisfies*

$$\begin{cases} \dfrac{\partial s_j}{\partial z} \in V_j \bigoplus V_{j+1} & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \dfrac{\partial s_k}{\partial z} \in V_m \bigoplus V_0 & \text{for} \quad k = m, \cdots, n, \end{cases}$$

and

$$\begin{cases} \Pi_{j+1}(\dfrac{\partial s_j}{\partial z}) \neq 0 & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \Pi_0(\dfrac{\partial^2 s_{m-1}}{\partial z^2}) \neq 0. \end{cases}$$

Proof. As before, write $s_j = e^a f_j \theta$ with $f_j = \sigma_j \theta_a^{-1}$, where $\sigma_j \in H^0(X, \mathcal{L}_j \otimes L(a))$.
[Case 1 : $j = 0, 1, \cdots, m-2$] By Lemma 5.3 we have $D_{\partial/\partial z}\sigma_j \in H^0(X, \mathcal{L}_j(P_0) \otimes L(a))$. Recall that if L is non-special then so is $L(P)$ for any point $P \in X$. Therefore, we see that $\mathcal{L}_j(P_0) \otimes L(a)$ is non-special by Corollary 5.1. Then, the Riemann-Roch formula implies that $h^0(X, \mathcal{L}_j(P_0) \otimes L(a)) = 2$. Now, obviously, $H^0(X, \mathcal{L}_j(P_0) \otimes L(a))$ is generated by σ_j, σ_{j+1} because $\mathcal{L}_j = \mathcal{L}_j(P_0) \otimes \mathcal{O}_X(-P_0)$ and $\mathcal{L}_{j+1} = \mathcal{L}_j(P_0) \otimes \mathcal{O}_X(-Q_0)$, which show that $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ are subsheaves of $\mathcal{L}_j(P_0)$. Therefore, we obtain

$$\frac{\partial s_j}{\partial z} \in V_j \bigoplus V_{j+1}.$$

Moreover, since $\iota_a^{-1}(\partial s_j/\partial z) = (\zeta^{-1} f_j + \partial f_j/\partial z)\theta_a$ and $\zeta^{-1} f_j \theta_a = \zeta^{-1}\sigma_j$ cannot be an element of $\Gamma(X_I, \mathcal{L}_j \otimes L(a))$, we must have $\Pi_{j+1}(\partial s_j/\partial z) \neq 0$.
[Case 2 : $j = m-1$] As in Case 1, we have $\partial s_{m-1}/\partial z \in V_{m-1} \bigoplus V_m$, $\iota_a^{-1}(\partial s_{m-1}/\partial z) \in H^0(X, \mathcal{L}_{m-1}(P_0) \otimes L(a))$ and $\Pi_m(\partial s_{m-1}/\partial z) \neq 0$. In this case, although $\mathcal{L}_m$ is not a subsheaf of $\mathcal{L}_{m-1}(P_0)$, it is enough to consider $\mathcal{L}_m(-\sum P_i)$, which is a subsheaf of $\mathcal{L}_m$. Next, we show $\Pi_0(\partial^2 s_{m-1}/\partial z^2) \neq 0$. We have

$$\iota_a^{-1}(\frac{\partial^2 s_{m-1}}{\partial z^2}) = (\zeta^{-2} f_{m-1} + 2\zeta^{-1}\frac{\partial f_{m-1}}{\partial z} + \frac{\partial^2 f_{m-1}}{\partial z^2})\theta_a \in H^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a)).$$

Notice that $\mathcal{L}_{m-1}(2P_0) = \mathcal{L}(P_0 - (m-1)Q_0 - \sum P_i)$ and $h^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a)) = 3$ by the Riemann-Roch formula. Hence, $H^0(X, \mathcal{L}_{m-1}(2P_0) \otimes L(a))$ has a section coming from a meromorphic section of $\mathcal{L} \otimes L(a)$ which has a pole of order 1 at P_0. In fact, $\zeta^{-2} f_{m-1}\theta_a$ is such a section. We observe that $\mathcal{L}_{m-1}, \mathcal{L}_m(-\sum P_i)$ and $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ are subsheaves of $\mathcal{L}_{m-1}(2P_0)$. Since $\lambda^{-1}\sigma_0$ is a section of $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ (notice that $(\lambda) = (m+1)P_0 - (m+1)Q_0$ on $U_0 \cup U_\infty$), it follows that $\mathcal{L}_{m-1}(2P_0)$ is generated by $\sigma_{m-1}, \sigma_m, \cdots, \sigma_n, \lambda^{-1}\sigma_0$. Among them, $\lambda^{-1}\sigma_0$ is the only one which has a pole of order 1 at P_0, which implies that $\Pi_0(\partial^2 s_{m-1}/\partial z^2) \neq 0$.
[Case 3 : $k = m, \cdots, n$] Similarly, we have $\iota_a^{-1}(\partial s_k/\partial z) \in H^0(X, \mathcal{L}_m(P_0) \otimes L(a))$ and $h^0(X, \mathcal{L}_m(P_0) \otimes L(a)) = n - m + 2$. In this case, $\mathcal{L}_m$ and $\mathcal{L}(P_0 - (m+1)Q_0 - \sum P_i)$ are subsheaves of $\mathcal{L}_m(P_0)$ and $\mathcal{L}_m(P_0)$ is generated by $\sigma_m, \cdots, \sigma_n, \lambda^{-1}\sigma_0$. Thus, we have $\partial s_k/\partial z \in V_m \bigoplus V_0$. q.e.d.

Now, we are in a position to prove the following theorem.

Theorem 5.1. *Let $l_0, \cdots, l_m (m \geq 2)$ be the subbundles of $\mathbf{R}^2 \times \mathbf{C}^{n+1}$ constructed above. Then l_0 determines a harmonic map $\psi_0 : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ of isotropy order m.*

Proof. Recall the map $\tau^1 : \Gamma(X_A, \mathcal{L}) \longrightarrow \mathbf{C}^{n+1}$. We see that $\tau^1(V_j) = l_j$. In fact, if we denote by $\{\sigma_i\}$ an orthonormal basis of $H^0(X, \mathcal{L})$ (which is independent of $z, \overline{z}$) then we have $s_j = \sum_{i=0}^n v_j^i(z, \lambda)\sigma_i$ and $\tau^1(s_j) = (v_j^0(z,1), \cdots, v_j^n(z,1)) \in l_j$, where we choose l_j corresponding to the choice of the orthonormal basis $\{\sigma_i\}$ of $H^0(X, \mathcal{L})$.

Therefore, the map τ^1 and the derivation $\partial/\partial z$ commute. Denote by $\pi_j : \mathbf{C}^{n+1} \longrightarrow l_j$ an orthogonal projection onto l_j. Then, we also have that $\tau^1 \circ \Pi_j = \pi_j \circ \tau^1$. It follows from Lemma 5.4 that

$$\begin{cases} \dfrac{\partial}{\partial z} l_j \subset l_j \oplus l_{j+1} & \text{for} \quad j = 0, 1, \cdots, m, \\ \pi_{j+1}(\dfrac{\partial l_j}{\partial z}) \neq 0 & \text{for} \quad j = 0, 1, \cdots, m-1, \\ \pi_0(\dfrac{\partial^2 l_{m-1}}{\partial z^2}) \neq 0, & \end{cases} \tag{5.3}$$

where we use the convention that $l_{m+1} = l_0$. Thus, a map $\psi = (l_0, l_1, \cdots, l_m) : \mathbf{R}^2 \longrightarrow F^m(\mathbf{C}P^n)$ is a primitive map. In fact, the complexification of the tangent bundle of $F^m(\mathbf{C}P^n)$ is given by $T^{\mathbf{C}}(F^m(\mathbf{C}P^n)) = \bigoplus_{i \neq j} \mathrm{Hom}(l_i, l_j)$. On the other hand, we have $(\psi^*\beta)(\partial/\partial z)$ has values in $\bigoplus_{i=0}^m \mathrm{Hom}(l_i, l_{i+1})$ with $l_{m+1} = l_0$ by (5.3). Since we are giving $F^m(\mathbf{C}P^n)$ the $(m+1)$- symmetric space structure such that $(l_j)_x$ is a ω^j- eigenspace of the automorphism τ_x of order $(m+1)$, where $\omega = \exp(2\pi\sqrt{-1}/(m+1))$, we have $[\mathcal{G}_1] = \bigoplus_{i=0}^m \mathrm{Hom}(l_i, l_{i+1})$ with $l_{m+1} = l_0$. Therefore, we see that ψ is a primitive map. Now, if $m \geq 2$ then $\varphi = \tilde{\pi} \circ \psi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ is a harmonic map, where $\tilde{\pi} : F^m(\mathbf{C}P^n) \longrightarrow \mathbf{C}P^n$ is the homogeneous projection. q.e.d.

When $m = 1$, we have a map $\psi : \mathbf{R}^2 \longrightarrow F^1(\mathbf{C}P^n) = \mathbf{C}P^n$. Since the condition of the primitivity of ψ is meaningless in this case, the above argument is not applied. In this case, we must calculate a holomorphic section of l_0 and investigate the divisor of the section. We omit the details. However, the following observation means that maps obtained in the both cases are harmonic maps of finite type :

Let $\{s_0, \cdots, s_n\}$ be an orthonormal basis for $H^0(X, \mathcal{L})$ and set $s_i(z) = \iota_a^{-1}(s_i)$. We have $s_i(0) = s_i$. Then $\{s_0(z), \cdots, s_n(z)\}$ is an orthonormal basis for $H^0(X, \mathcal{L} \otimes L(a))$. Define $F(z, \lambda)$, which depends only on λ, z, by

$$(s_0(z), \cdots, s_n(z)) = (s_0, \cdots, s_n)F(z, \lambda). \tag{5.4}$$

Let $f(\zeta)$ be any regular algebraic function of ζ on X_A. Define $Y(z, \lambda)$ by

$$(s_0(z), \cdots, s_n(z))Y(z, \lambda) = f(\zeta)(s_0(z), \cdots, s_n(z)).$$

Then we have

$$\begin{aligned}(s_0, \cdots, s_n)F(z,\lambda)Y(z,\lambda) &= (s_0(z), \cdots, s_n(z))Y(z,\lambda) \\ &= f(\zeta)(s_0(z), \cdots, s_n(z)) \\ &= f(\zeta)(s_0, \cdots, s_n)F(z,\lambda) \\ &= (s_0, \cdots, s_n)Y(0,\lambda)F(z,\lambda).\end{aligned}$$

Therefore, we obtain $Y(z,\lambda) = \mathrm{Ad}F(z,\lambda)^{-1} \cdot Y(0,\lambda)$. Differentiating this equation, we obtain $dY = [Y, F^{-1}dF]$, i.e., $Y(z,\lambda)$ is a polynomial Killing field. Thus, the corresponding map is a primitive harmonic map of finite type by the results of [BFPP] and [Bu] and our harmonic map $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ is given by the first column vector of $F(z,1)$, which is a framing of φ.

To obtain an explicit form of the harmonic map φ or the framing $F(z,1)$ constructed in the above, we may use the equation (5.4). Let $\{\zeta_0, \zeta_1, \cdots, \zeta_n\}$ be the elements of $\pi^{-1}(1)$. We can evaluate the equation (5.4) at $\zeta = \zeta_0, \zeta_1, \cdots, \zeta_n$ with the same $F(z,1)$. Therefore, we have

$$\begin{pmatrix} s_0(z)\mid_{\zeta_0} & \cdots & s_n(z)\mid_{\zeta_0} \\ \vdots & \ddots & \vdots \\ s_0(z)\mid_{\zeta_n} & \cdots & s_n(z)\mid_{\zeta_n}\end{pmatrix} = \begin{pmatrix} s_0\mid_{\zeta_0} & \cdots & s_n\mid_{\zeta_0} \\ \vdots & \ddots & \vdots \\ s_0\mid_{\zeta_n} & \cdots & s_n\mid_{\zeta_n}\end{pmatrix} F(z,\lambda). \tag{5.5}$$

Let $S_0, \cdots, S_n$ be a local frame around $\pi^{-1}(1)$ which has the properties $S_i(\zeta_j) = 0$ for $i \neq j$ and $f \cdot S_i(\zeta_i)\overline{\rho_{X*}S_i(\zeta_i)} = 1$. If we express s_i around $\pi^{-1}(1)$ as

$$s_i = \sum_{j=0}^{n} v_{ij} S_j,$$

then we see that

$$\begin{aligned}\delta_{ik} &= h(s_i, s_k) \\ &= \sum_{j=0}^{n}(f \cdot \sum_l v_{il}S_l(\zeta_j)\overline{\sum_m v_{km}S_m(\rho_{X*}\zeta_j)}) \\ &= \sum_j^n v_{ij}\overline{v_{kj}},\end{aligned}$$

which shows that the matrix $M = (M_{ij})$ with $M_{ij} = s_j\mid_{\zeta_i}$ is non-singular. Thus we obtain a formula $F(z,1) = M^{-1} \cdot M(z)$.

Now, we give some examples

Example 5.1. Consider $\pi : \mathbf{C}P^1_\zeta \longrightarrow \mathbf{C}P^1_\lambda$ defined by $\zeta \mapsto \lambda = \zeta^{-3}$. Then $(\pi) = 3(\infty) - 3(0)$ and $R = 2(\infty) + 2(0)$, we must regard $P_0 = \{\zeta = \infty\}$, $Q_0 = \{\zeta = 0\}$. Define

$\mathcal{L}$ by $\mathcal{L} = \mathcal{O}_X(2(\infty))$. Then we have

$$\begin{cases} \mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_X(-2P_0) = \mathcal{O}_X, \\ \mathcal{L}_1 = \mathcal{L} \otimes \mathcal{O}_X(-P_0 - Q_0) = \mathcal{O}_X((\infty) - (0)), \\ \mathcal{L}_2 = \mathcal{L} \otimes \mathcal{O}_X(-2Q_0) = \mathcal{O}_X(2(\infty) - 2(0)), \end{cases}$$

which implies that we may set $s_0 = 1, s_1 = \zeta, s_2 = \zeta^2$ and $s_0(z) = e^a, s_1(z) = e^a\zeta, s_2(z) = e^a\zeta^2$, where $a = z\zeta - \overline{z}\zeta^{-1}$. Since $\pi^{-1}(1) = \{1, \omega, \omega^2\}$, where $\omega = (-1 - \sqrt{-3})/2$, we have

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad M(z) = \begin{pmatrix} e^{z-\overline{z}} & e^{z-\overline{z}} & e^{z-\overline{z}} \\ e^{z\omega - \overline{z\omega}} & e^{z\omega - \overline{z\omega}}\omega & e^{z\omega - \overline{z\omega}}\omega^2 \\ e^{z\omega^2 - \overline{z\omega^2}} & e^{z\omega^2 - \overline{z\omega^2}}\omega^2 & e^{z\omega^2 - \overline{z\omega^2}}\omega \end{pmatrix}.$$

Therefore, we can easily obtain $F(z, 1) = M^{-1} \cdot M(z)$ and we see that $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^2$ is a harmonic map of isotropy order 2. Note that φ is doubly periodic. We recommend the readers to obtain the explicit form of $F(z, 1)$. In the same way, we may calculate $F(z, 1)$ in the case where π is given by $\pi(\zeta) = \zeta^{-(n+1)}$, where we choose $\mathcal{L} = \mathcal{O}_X(n(\infty))$ and obtain a harmonic map $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ of isotropy order n (cf. [T1]), which is doubly periodic for $n = 1, 2, 3, 5$. The case of $\pi(\zeta) = \zeta^{n+1}$ is also similar, where we choose $\mathcal{L} = \mathcal{O}_X(n(0))$ and we see that $1, \zeta^{-1}, \cdots, \zeta^{-n}$ are global sections, respectively, of $\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_n$.

Example 5.2. Consider $\pi : X \longrightarrow \mathbf{C}P^1_\lambda$ defined by $\pi(\zeta) = \frac{1}{\alpha}\zeta^3\frac{(\zeta-\alpha)}{(\zeta-\alpha^{-1})} = \lambda$ as in Example 3.2, where we suppose that $0 < \alpha < 1$ for simplicity. Then we have $(\pi) = 3(0) + (\alpha) - 3(\infty) - (\alpha^{-1})$, $R = 2(0) + (p) + (p^{-1}) + 2(\infty)$ and $X = \mathbf{C}P^1_\zeta$ as in Example 3.2. In this case, $P_0 = \{\zeta = 0\}, P_1 = \{\zeta = \alpha\}, Q_0 = \{\zeta = \infty\}$ and $Q_1 = \{\zeta = \alpha^{-1}\}$. Define $\mathcal{L} = \mathcal{O}_X(3(0))$. Then $f : \mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \longrightarrow \mathcal{O}_X(R)$ is given by

$$f = \frac{-\zeta}{(\zeta - p)(\zeta - p^{-1})},$$

which is non-negative on $X_{\mathbf{R}}$. We have

$$\begin{cases} \mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_X(-2P_0 - P_1) = \mathcal{O}_X((0) - (\alpha)), \\ \mathcal{L}_1 = \mathcal{L} \otimes \mathcal{O}_X(-P_0 - Q_0 - P_1) = \mathcal{O}_X(2(0) - (\infty) - (\alpha)), \\ \mathcal{L}_2 = \mathcal{L} \otimes \mathcal{O}_X(-2Q_0) = \mathcal{O}_X(3(0) - 2(\infty)). \end{cases}$$

Set

$$\begin{cases} s_0 = \dfrac{1}{\sqrt{3\alpha}}\dfrac{\zeta - \alpha}{\zeta}, \quad s_1 = \dfrac{1}{\sqrt{3\alpha}}\dfrac{\zeta - \alpha}{\zeta^2}, \\ s_2 = \sqrt{\dfrac{1 - \alpha^2}{3\alpha}}\dfrac{1}{\zeta^2}, \quad s_3 = \sqrt{\dfrac{1}{3\alpha}}\dfrac{(1 - \alpha\zeta)}{\zeta^3}. \end{cases}$$

Then we see that s_0 is a global section of $\mathcal{L}_0$, s_1 a global section of $\mathcal{L}_1$ and s_2, s_3 global sections of $\mathcal{L}_2$. Evaluating $h(s_i, s_j)$ at $\lambda = 0$ $(\pi^{-1}(0) = \{0, 0, 0, \alpha\})$, we easily see that

$\{s_0, \cdots, s_3\}$ is orthonormal basis of $H^0(X, \mathcal{L})$. Let $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ be the elements of $\pi^{-1}(1)$. Notice that η_0, η_1 are solutions of the equation $\zeta + 1/\zeta = (\alpha + \sqrt{\alpha^2+8})/2$ and η_2, η_3 are solutions of the equation $\zeta + 1/\zeta = (\alpha - \sqrt{\alpha^2+8})/2$. We also see that $\{e^a s_0, \cdots, e^a s_3\}$ is orthonormal basis of $H^0(X, \mathcal{L} \otimes L(a))$. Now, $F(z,1) = M^{-1} \cdot M(z)$ is computable. Obtaining the explicit form of $F(z,1)$ is left to the readers.

Example 5.3. We will give an example of which the spectral curve is an elliptic curve. First of all, we will address some fundamental facts on elliptic functions.
[*Weierstrass zeta-function*]
Let $\mathbf{L} = \mathbf{Z} \oplus \tau\mathbf{Z}$, where τ is a complex number with $\mathrm{Im}(\tau) > 0$. The *Wierstrass zeta-function* $\zeta_w(u)$ is defined by

$$\zeta_w(u) = \frac{1}{u} + \sum_{\omega \in \mathbf{L} \setminus (0,0)} \{ \frac{1}{(u-\omega)} + \frac{u}{\omega^2} + \frac{1}{\omega} \},$$

which has a pole of order 1 at $u = 0$. Set

$$\mathcal{P}(u) = -\frac{d}{du}\zeta_w(u),$$

which uniformly converges on each compact subset and called *Weierstass* $\mathcal{P}$*−function.* We have

$$\frac{d}{du}\mathcal{P}(u) = -2\sum_{\omega \in \mathbf{L}} \frac{1}{(u-\omega)^3}.$$

The definition of the summation means that $\frac{d}{du}\mathcal{P}(u)$ is invariant under the translations $u \to u+1$ and $u \to u+\tau$. Hence, $\frac{d}{du}\mathcal{P}(u)$ is doubly-periodic function. Therefore we may set

$$\begin{cases} \mathcal{P}(u+1) - \mathcal{P}(u) = c_1 \\ \mathcal{P}(u+\tau) - \mathcal{P}(u) = c_2 \end{cases}$$

where c_1, c_2 are some complex numbers. On the other hand, since $\mathcal{P}(u)$ is obviously even function, by setting $u = -1/2$ or $u = -\tau/2$ in the above equations we have $c_1 = c_2 = 0$. Therefore we see that the Weierstrass $\mathcal{P}-$function is doubly-periodic with periods $1, \tau$. Integrating $\mathcal{P}-$function, we have

$$\begin{cases} \zeta_w(u+1) - \zeta_w(u) = A \\ \zeta_w(u+\tau) - \zeta_w(u) = B \end{cases}$$

where A, B are some complex numbers. Notice that the residue theorem yields that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial P_a} \zeta_w(u) du = 1$$

where P_a is some fundamental domain of the torus $\mathbf{R}^2/\mathbf{L}$. The integration of $\zeta_w(u)$ on P_a turns out to be $A\tau - B$, which yields, so-called, *Legendre's relation* :

$$A\tau - B = 2\pi\sqrt{-1}.$$

[*Jacobi's 1st theta function*]
Let $p(u) = \exp(\pi\sqrt{-1}u), q = \exp(\pi\sqrt{-1}\tau)$. Then the *Jacobi's 1st theta function* $\theta_1(u)$ is defined by

$$\theta_1(u) = \sqrt{-1}\sum_{n\in\mathbf{Z}}(-1)^n p(u)^{2n-1}q^{(n-\frac{1}{2})^2}.$$

By changing $n \to -n+1$ in the summation we see that $\theta_1(u)$ is an odd function. In particular, we have $\theta_1(0) = 0$. Moreover, since $(n-\frac{1}{2})^2 + (2n-1) = (n+\frac{1}{2})^2 - 1$, the definition of the summation gives the following relations :

$$\text{(5.6)} \qquad \begin{cases} \theta_1(u+1) = -\theta_1(u) \\ \theta_1(u+\tau) = -p(u)^{-2}q^{-1}\theta_1(u) \end{cases}$$

With these facts in mind, we may construct a meromorphic function on some elliptic curve.

Let $X = \mathbf{R}^2/\mathbf{L}$ be a two-torus with lattice $\mathbf{L} = \mathbf{Z} \oplus \sqrt{-1}t\mathbf{Z}$, where t is some positive real number. In this case, we have $\overline{\zeta_w(u)} = \zeta_w(\overline{u})$. Define a function $\psi(z,\overline{z},u)$ on X by

$$\text{(5.7)} \qquad \psi(z,\overline{z},u) = \exp((\zeta_w(u-P_0) - Au)z - (\zeta_w(u-Q_0) - Au)\overline{z})\times$$
$$\frac{\theta_1(u-F_1)\cdots\theta_1(u-F_k)\theta_1(u-P_0)^m\theta_1(u-P_1)\cdots\theta_1(u-P_{n-m})\theta_1(u-G-z+\overline{z})}{\theta_1(u-E_1)\theta_1(u-E_2)\cdots\theta_1(u-E_{n+k+1})},$$

where

$$\begin{cases} \mathcal{L} \cong \mathcal{O}_X(D), \qquad D = \displaystyle\sum_{i=1}^{n+k+1} E_i - \sum_{i=1}^{k} F_i, \\ G = D - mP_0 - \displaystyle\sum_{i=1}^{n-m} P_i. \end{cases}$$

Hence, $\mathcal{L}$ is a divisor line bundle of degree $(n+1)$. It follows from (5.6) that $\psi(z,\overline{z},u+1) = \psi(z,\overline{z},u), \psi(z,\overline{z},u+\sqrt{-1}t) = \psi(z,\overline{z},u)$, i.e., $\psi(z,\overline{z},u)$ is a meromorphic function on X with fixed $z,\overline{z}$. Moreover, since ψ behaves like $\exp(z\zeta^{-1} + O(\zeta))$ near P_0 and behaves like $\exp(-\overline{z}\zeta + O(1/\zeta))$ near Q_0 and ψ has a divisor $-D$ on $X \setminus \{P_0, Q_0\}$, we see that $\psi(z,\overline{z},u)\theta_A$ belongs to $H^0(X, \mathcal{O}_X(D)\otimes L(a))$(see [T2]).

Now, consider a function

$$g(u) = \exp(4\pi\sqrt{-1}u)\Big(\frac{\theta_1(u-R_1)}{\theta_1(u-R_2)}\Big)^4$$

on X, where $R_1 = \sqrt{-1}/4, R_2 = 3\sqrt{-1}/4 = \rho_X(R_1)$ and $\rho_X(P) = \overline{P} \mod \mathbf{L}$. It follows from (5.6) that $g(u)$ is a meromorphic function on X. Define a covering map $\pi : X \longrightarrow \mathbf{C}P^1$ by $\pi(u) = g(u)/g(\sqrt{-1}/2)$. Then we have

$$\begin{cases} (\pi) = 4(R_1) - 4(R_2) \\ \quad R = 3(R_1) + (R_3) + (\rho_X(R_3)) + 3(R_2) \end{cases}$$

for some point $R_3 \in X \backslash X_{\mathbf{R}}$. Therefore, we have $P_0 = R_1, Q_0 = R_2$. Let $\mathcal{L} = \mathcal{O}_X(3R_1 + R_3)$ be a line bundle over X of degree 4. We see that $\pi^{-1}(1) = \{0, 1/2, \sqrt{-1}/2, 1/2 + \sqrt{-1}/2\}$. It follows that each point of $\pi^{-1}(1)$ is fixed by the real structure ρ_X. In this case, we have $\mathcal{L}_0 = \mathcal{O}_X(R_3), \mathcal{L}_1 = \mathcal{O}_X(R_1 + R_3 - R_2), \mathcal{L}_2 = \mathcal{O}_X(2R_1 + R_3 - 2R_2), \mathcal{L}_3 = \mathcal{O}_X(3R_1 + R_3 - 3R_2)$. Set $\eta_0 = 0, \eta_1 = 1/2, \eta_2 = \sqrt{-1}/2, \eta_3 = 1/2 + \sqrt{-1}/2$. By choosing some constants c_0, c_1, c_2, c_3, we may define an orthonormal basis $\{s_i\}$ of $H^0(X, \mathcal{L})$ by

$$s_i = c_i \frac{\theta_1(u - \eta_0) \cdots \theta_1(u - \hat{\eta_i}) \cdots \theta_1(u - \eta_3)}{\theta_1(u - R_1)^3 \theta_1(u - R_3)},$$

where $\hat{\eta_i} = 3R_1 + R_3 - (\eta_0 + \cdots \eta_{i-1} + \eta_{i+1} + \cdots + \eta_3)$. In fact, it follows from the positive definiteness of h that $\hat{\eta_i} \neq \eta_i$. Now, we set

$$\begin{cases} s_0(z) = \dfrac{1}{c_0} \dfrac{\theta_1(u - R_1)^3 \theta_1(u - R_3 - z - \overline{z})}{\theta_1(u - \hat{\eta_0})\theta_1(u - \eta_1)\theta_1(u - \eta_2)\theta_1(u - \eta_3)} \exp(f(z, u)), \\ s_1(z) = \dfrac{1}{c_1} \dfrac{\theta_1(u - R_1)^2 \theta_1(u - R_2)\theta_1(u - R_1 - R_3 + R_2 - z - \overline{z})}{\theta_1(u - \eta_0)\theta_1(u - \hat{\eta_1})\theta_1(u - \eta_2)\theta_1(u - \eta_3)} \exp(f(z, u)), \\ s_2(z) = \dfrac{1}{c_2} \dfrac{\theta_1(u - R_1)\theta_1(u - R_2)^2 \theta_1(u - 2R_1 - R_3 + 2R_2 - z - \overline{z})}{\theta_1(u - \eta_0)\theta_1(u - \eta_1)\theta_1(u - \hat{\eta_2})\theta_1(u - \eta_3)} \exp(f(z, u)), \\ s_3(z) = \dfrac{1}{c_3} \dfrac{\theta_1(u - R_2)^3 \theta_1(u - 3R_1 - R_3 + 3R_2 - z - \overline{z})}{\theta_1(u - \eta_0)\theta_1(u - \eta_1)\theta_1(u - \eta_2)\theta_1(u - \hat{\eta_3})} \exp(f(z, u)), \end{cases}$$

where $f(z, u) = \exp((\zeta_w(u - R_1) - Au)z - (\zeta_w(u - R_2) - Au)\overline{z})$. Then $F(z, 1) = M^{-1} \cdot M(z)$ is computable and the first column vector of $F(z, 1)$ gives a harmonic map of $\mathbf{R}^2$ into $\mathbf{C}P^3$ with isotropy order 3(i.e., superconformal harmonic map).

We will discuss the double periodicity of our harmonic map $\mathbf{R}^2 \longrightarrow \mathbf{C}P^n$ of isotropy order m and the construction of spectral data from them elsewhere.

References

[B] M.Black, *Harmonic maps into homogeneous spaces*, Pitman Res. Notes in Math., vol. **255**, Longman, Harlow, 1991.

[Bu] F.E.Burstall, *Harmonic tori in spheres and complex projective spaces*, J.Reine Angew. Math. **469**(1995), 149-177.

[BFPP] F.E.Burstall, D.Ferus, F.Pedit and U.Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. **138**(1993), 173-212.

[BP] F.E.Burstall and F.Pedit, *Harmonic maps via Adler-Kostant-Symes theory*, Harmonic maps and Integrable Systems (A.P.Fordy and J.C.Wood, eds.), Aspect of Mathematics E23, Vieweg, 1994, pp221-272.

[H] N.J.Hitchin, *Integrable systems in Riemannian geometry*, preprint.

[K] O.Kowalski, *Generalized Symmetric Spaces*, Lect. Notes in Math. **805**, Springer-Verlag, 1980.

[Mc1] I.McIntosh, *A construction of all non-isotropic harmonic tori in complex projective space*, International J. Math. **6**(1995), 831-879.

[Mc2] I.McIntosh, *Two remarks on the construction of harmonic tori in* $\mathbf{C}P^n$, International J. Math. **7**(1996), 515-520.

[T1] T.Taniguchi, *Non-isotropic harmonic tori in complex projective space and configurations of points on the rational curve*, preprint.

[T2] T.Taniguchi, *Non-isotropic harmonic tori in complex projective space and configurations of points on elliptic curves, preprint.*