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CLASSIFICATION OF PLURIHARMONIC MAPS FROM COMPACT COMPLEX MANIFOLDS WITH POSITIVE FIRST CHERN CLASS INTO COMPLEX GRASSMANN MANIFOLDS

Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

SEIICHI UDAGAWA

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Abstract. We prove that any pluriharmonic map from a compact complex manifold with positive first Chern class (defined outside a certain singularity set of codimension at least two) into a complex Grassmann manifold of rank two is explicitly constructed from a rational map into a complex projective space. Under some restrictions on dimension and rank of the domain manifold and the target manifold, respectively, we also prove that similar results hold for other complex Grassmann manifolds as targets.

Introduction. Let $\varphi: M \to N$ be a smooth map from a complex manifold into a Riemannian manifold. Then, φ is said to be *pluriharmonic* if the (0, 1)-exterior covariant derivative $D''\partial\varphi$ of the $(1, 0)$-differential $\partial\varphi$ of φ vanishes identically. Let ∇^{φ} be the pull-back connection on the pull-back bundle $\varphi^{-1}TN$. We have

$$(0.1) \qquad (D''\partial\varphi)(\bar{X}, Y) = \nabla^{\varphi}_{\bar{X}}\partial\varphi(Y) - \partial\varphi(\bar{\partial}_{\bar{X}}Y), \qquad X, Y \in C^{\infty}(TM^{1,0}),$$

where $TM^{1,0}$ is the holomorphic tangent bundle of M. If $\varphi^{-1}TN^{\boldsymbol{C}}$ has the Koszul-Malgrange holomorphic structure, that is, the (0, 1)-part of ∇^{φ} coincides with the $\bar{\partial}$-operator, we may say that φ is pluriharmonic if and only if φ sends any holomorphic section of $TM^{1,0}$ to a holomorphic section of $\varphi^{-1}TN^{\boldsymbol{C}}$. It is easily seen that if φ is holomorphic and N is a Kähler manifold then $\varphi^{-1}TN^{1,0}$ has the Koszul-Malgrange holomorphic structure, hence any holomorphic map is pluriharmonic. Note that an anti-holomorphic map is also pluriharmonic if N is a Kähler manifold. Conversely, the existence of the Koszul-Malgrange holomorphic structure on $\varphi^{-1}TN^{\boldsymbol{C}}$ is ensured if φ is pluriharmonic and N has nonnegative or nonpositive curvature operator. In this case, if N is a Kähler manifold, then $\varphi^{-1}TN^{1,0}$ has the Koszul-Malgrange holomorphic structure (cf. [O-U2]). From the point of view of Riemannian geometry, the most interesting property of pluriharmonic maps is that it

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is a harmonic map with respect to any Kähler metric on M. Therefore, the concept of pluriharmonic maps generalizes that of harmonic maps for Riemann surfaces. Moreover, when one restricts a pluriharmonic map from M to any holomorphic curve C of M, it induces a harmonic map from C into N. A natural question is: Which class of pluriharmonic maps comes from holomorphic maps? This question is treated in [O-U1]. As a special case, if the target is a complex Grassmann manifold $G_k(\boldsymbol{C}^n)$ of k-dimensional complex linear subspaces in $\boldsymbol{C}^n$, any pluriharmonic map φ from a Kähler manifold M is $\pm$-holomorphic provided the rank of $d\varphi$ over $\boldsymbol{R}$ is greater than or equal to $2(n-k-1)(k-1)+3$, where a map is said to be $\pm$-holomorphic if it is holomorphic or anti-holomorphic. In the case of an M with $c_1(M)>0$ and $b_2(M)=1$, the rank condition on φ may be replaced by $\dim_{\boldsymbol{C}} M \geq (n-k-1)(k-1)+2$ and this dimension estimate is best possible. In fact, there are so many examples of pluriharmonic maps which are not $\pm$-holomorphic (see [O-U1]). Then, the following problem arises: Classify all pluriharmonic maps which are not $\pm$-holomorphic. In the following, we restrict our attention to the complex Grassmann manifolds as the target manifold. In the case where the domain is the Riemann sphere, this problem was treated and solved by several authors [Rm], [C-W], [B-W], [B-S], [Wol], [Wd1], who proved that *any harmonic map from the Riemann sphere S^2 into $G_k(\boldsymbol{C}^n)$ may be constructed from a holomorphic map $S^2 \to G_t(\boldsymbol{C}^n)$ for some $1 \leq t \leq k$*. This result originates from the work of Burns [Bn], Din-Zakrewski [D-Z], Glaser-Stora [G-S] and Eells-Wood [E-W] with a complex projective space as target. Given a map $\varphi: M \to G_k(\boldsymbol{C}^n)$, we may identify φ with the pull-back of the universal bundle over $G_k(\boldsymbol{C}^n)$ by φ, denoted by $V(\varphi)$, which is a complex subbundle of the trivial bundle $M \times \boldsymbol{C}^n$. We have a sequence of the *∂'-Gauss bundles* by taking the image of the $(1, 0)$-part of the second fundamental form of each subbundle. Wolfson proved that this sequence must terminate if $M=S^2$. In general, φ has intersection with certain ∂'-Gauss bundle, say the $(r+1)$-th Gauss bundle, and the least such integer r is called the *∂'-isotropy order* of φ. A holomorphic map has *infinite* ∂'-isotropy order, hence one tries to increase the ∂'-isotropy order of a given map by certain algebraic replacement, which is called the *forward replacement*. It is known that Wolfson's harmonic map sequence can be obtained by a successive application of the forward replacements (for details, see Section 2). This is the method of Burstall-Wood. In particular, they proved that any harmonic map of *finite* ∂'-isotropy order from S^2 into $G_k(\boldsymbol{C}^n)$ with $k=2, 3, 4, 5$ may be obtained by a successive application of the *backward* replacement from a holomorphic map $S^2 \to G_t(\boldsymbol{C}^n)$ with $1 \leq t \leq k-1$, where the backward replacement is the inverse procedure of the forward replacement (see Section 2). Note that the case of infinite isotropy order is rather easy to treat for any k. For higher dimensional domains, we assume that M is a compact complex manifold with positive first Chern class, denoted by $c_1(M)>0$. However, there are many difficulties. For example, the ∂'-Gauss bundle of φ has non-removable singularities, and its rank may be greater than that of $V(\varphi)$. Therefore, it seems to be impossible to generalize Wolfson's method to higher dimensional case. On the other hand, Ohnita and the present author [O-U2] treated a part of the problem using Burstall-Wood's method and proved that any pluriharmonic map φ from $M \setminus S_\varphi$ with M as above into $G_k(C^n)$ may be obtained from a rational map f: $M \to G_t(C^n)$ for some t provided (1) $k=1$, or (2) $k=2, 3$ and $n \leq 12$, where S_φ is a certain singularity set of codimension at least two (see Section 2). As in the case of harmonic maps from the Riemann sphere, a pluriharmonic map with infinite isotropy order is easier to treat (see Proposition 4.3). Note that any pluriharmonic map from $M \setminus S_\varphi$ with M and S_φ as above into a complex projective space CP^{n-1} with the Fubini-Study metric has infinite isotropy order, which is the reason why there is no restriction on n for $G_1(C^n) = CP^{n-1}$ in the result stated above. Moreover, in this case, the uniqueness of the sequence of pluriharmonic maps is also ensured. Even if a pluriharmonic map $\varphi : M \to G_k(C^n)$ has singularities, the nilpotency of certain End$(V(\varphi))$-valued holomorphic differentials is preserved, hence we may apply their method to increase the ∂'-isotropy order by one, that is, the first step of the procedure is the same as in the case of $M = S^2$. However, in higher dimension, the next step of the procedure cannot be applied because the situation rank $G'(\varphi) >$ rank $V(\varphi)$ may occur. This is the main reason for the restriction on n in the result stated above.

The main purpose of this paper is to prove that *any pluriharmonic map* $\varphi : M \setminus S_\varphi \to G_2(C^n)$ *with finite* ∂''*-isotropy order may be obtained by a successive application of the forward replacement and extension from a pluriharmonic map* $\varphi^0 : M \setminus S_{\varphi^0} \to CP^{n-1}$ (Theorem 4.2). This technique is partially applied to the case where the target is $G_k(C^n)$ with $k=3, 4$, and similar results with restriction on n are also true (see Section 6).

In Section 5, we give some examples of pluriharmonic maps of CP^2 which has finite ∂'-isotropy order.

We refer the reader to [E-L] for recent developments on harmonic map theory, to [B-B-B-R], [B-B], [O-U1,2], [Ud] for the stability and complex-analyticity of pluriharmonic maps, and to [B-R], [Uh], [V], [Wd2] for the construction of harmonic maps from the Riemann sphere to Lie groups. Finally, we mention that Ohnita and Valli [O-V] generalized the results of [Uh], [V] to the class of meromorphically pluriharmonic maps. Their assumption on the domain manifold M is slightly weaker than that of ours. On the other hand, when $c_1(M) > 0$, our class of pluriharmonic maps is slightly wider than that of theirs. In the case of Lie groups as targets, there is a similar concept called *basic transform* corresponding to our forward (or backward) replacement. We remark that the method using the basic transform is not established yet, and that even if it is established our results are not covered by it (cf. [Wd1]).

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many valuable suggestions.

1. Preliminaries. Let E be a unitary vector bundle over a complex manifold M, that is, E is endowed with a Hermitian fibre metric h and a connection ∇^E compatible with h. Let F be a complex subbundle of E and let S be the Hermitian orthogonal complement of F in E with respect to h. Then, F and S also become unitary vector bundles with respect to the induced Hermitian structures. The second fundamental forms, $A^{S,F}$ and $A^{F,S}$, are defined by

$$\nabla^E_X v = \nabla^F_X v + A^{F,S}_X(v)\,, \qquad \nabla^E_X w = \nabla^S_X w + A^{S,F}_X(w) \tag{1.1}$$

for any $X \in C^\infty(TM)$, $v \in C^\infty(F)$, $w \in C^\infty(S)$, where ∇^E, ∇^F and ∇^S are the Hermitian connections of E, F and S, respectively, and $A^{F,S}$ (resp. $A^{S,F}$) is regarded as a Hom(F, S)-valued (resp. Hom(S, F)-valued) 1-form on M. We easily obtain

$$A^{F,S} = -(A^{S,F})^*\,, \tag{1.2}$$

where $(\ )^*$ denotes the adjoint of $(\ )$ with respect to h. By the complex structure of M, we may decompose $A^{F,S}$ as $A^{F,S} = A^{F,S}_{(1,0)} + A^{F,S}_{(0,1)}$. Let D be the exterior covariant differentiation defined by the induced connection on Hom(F, S), and D', D'' the (1,0)- and (0,1)-parts of D, that is, $D = D' + D''$. The (0,1)-exterior covariant derivative $D''A^{F,S}_{(1,0)}$ of $A^{F,S}_{(1,0)}$ is defined by

$$(D''A^{F,S}_{(1,0)})(\bar{Z}, W) = \nabla^S_{\bar{Z}} \circ A^{F,S}_W - A^{F,S}_W \circ \nabla^F_{\bar{Z}} - A^{F,S}_{\partial_{\bar{Z}} W}\,, \tag{1.3}$$

where $Z, W \in C^\infty(TM^{1,0})$. $D'A^{F,S}_{(0,1)}$ is defined similarly. Now, assume that E has the Koszul-Malgrange holomorphic structure, that is, a holomorphic structure compatible with the Hermitian structure of E, and that F is a holomorphic subbundle of E. We may endow S with a holomorphic vector bundle structure by the isomorphism $S \cong E/F$, which is, in fact, nothing but the Koszul-Malgrange holomorphic structure (cf. [B-S]). Then, Hom(F, S) also has the Koszul-Malgrange holomorphic structure and a smooth section A of $T^*M^{1,0} \otimes \mathrm{Hom}(F, S)$ is said to be *holomorphic* if $D''A \equiv 0$.

Let $\varphi : M \to G_k(\boldsymbol{C}^n)$ be a smooth map from a complex manifold into a complex Grassmann manifold of k-dimensional complex linear subspaces in $\boldsymbol{C}^n$. Then, we may identify φ with a complex subbundle $V(\varphi)$ of rank k of the trivial bundle $V(\boldsymbol{C}^n) = M \times \boldsymbol{C}^n$, of which the fibre at $x \in M$ is given by $\varphi(x)$. Note that $V(\varphi)$ is the pull-back of the universal bundle T over $G_k(\boldsymbol{C}^n)$ by φ.

For any complex subbundle E of $V(\boldsymbol{C}^n)$, we denote by $E^\perp$ the Hermitian orthogonal complement of E in $V(\boldsymbol{C}^n)$ with respect to the standard Hermitian fibre metric on $V(\boldsymbol{C}^n)$. Moreover, for any complex subbundle F of E, we denote by $E \ominus F$ the Hermitian orthogonal complement of F in E, that is, $E \ominus F = E \cap F^\perp$.

Set

$$A^\varphi = A^{V(\varphi), V(\varphi)^\perp}\,, \qquad A^{\varphi^\perp} = A^{V(\varphi)^\perp, V(\varphi)}\,, \tag{1.4}$$

where $\varphi^{\perp}$ is a map from M into $G_{n-k}(\boldsymbol{C}^n)$ and $V(\varphi^{\perp})=V(\varphi)^{\perp}$. Then, by (1.2) we obtain

$$A^{\varphi}_{(1,0)}=-(A^{\varphi^{\perp}}_{(0,1)})^*\,, \qquad A^{\varphi}_{(0,1)}=-(A^{\varphi^{\perp}}_{(1,0)})^*\,. \tag{1.5}$$

The property of φ may be interpreted in terms of the property of A^{φ}. In fact, we have:

PROPOSITION 1.1 (cf. [O-U2]). (I) *The following statements are mutually equivalent*:

(1) *φ is holomorphic (resp. anti-holomorphic).*

(2) *$V(\varphi)$ is a holomorphic (resp. an anti-holomorphic) subbundle of $V(\boldsymbol{C}^n)$.*

(3) *$A^{\varphi}_{(0,1)}\equiv 0$ (resp. $A^{\varphi}_{(1,0)}\equiv 0$).*

(II) *φ is pluriharmonic if and only if $D''A^{\varphi}_{(1,0)}\equiv 0$, or equivalently $D'A^{\varphi}_{(0,1)}\equiv 0$.*

(III) *φ is pluriharmonic if and only if $\varphi^{\perp}$ is pluriharmonic.*

In fact, we may say that if φ is pluriharmonic then $A^{\varphi}_{(1,0)}$ is a holomorphic section of $T^*M^{1,0}\otimes \mathrm{Hom}(V(\varphi), V(\varphi^{\perp}))$ by Proposition 1.1, (II) and the following fact:

PROPOSITION 1.2 (cf. [O-U2]). *If φ is pluriharmonic, each of $V(\varphi)$ and $V(\varphi^{\perp})$ has the Koszul-Malgrange holomorphic structure. In particular, any holomorphic subbundle of $V(\varphi)$ or $V(\varphi^{\perp})$, and its Hermitian orthogonal complement in $V(\varphi)$ or $V(\varphi^{\perp})$ have the Koszul-Malgrange holomorphic structures.*

It follows from Propositions 1.1 and 1.2 that if φ is pluriharmonic, then $A^{\varphi^{\perp}}_{(1,0)}$ is also a holomorphic section of $T^*M^{1,0}\otimes \mathrm{Hom}(V(\varphi^{\perp}), V(\varphi))$.

2. A general construction of pluriharmonic maps.

Let $\varphi: M\to G_k(\boldsymbol{C}^n)$ be a pluriharmonic map from a complex manifold. A general theory for the construction of pluriharmonic maps is quite similar to the one of harmonic maps for Riemann surfaces except that non-removable singularities appear. Here, we review the construction (cf. [Wd1], [B-W], [O-U2]).

PROPOSITION 2.1. *Define $\tilde{\varphi}$ by*

$$V(\tilde{\varphi})=(V(\varphi)\ominus\alpha)\oplus\beta\,, \tag{2.1}$$

where α and β satisfy the following conditions (1), (2):

(1) *α and β are holomorphic subbundles of $V(\varphi)$ and $V(\varphi^{\perp})$, respectively,*

(2) $A^{\varphi}_{(1,0)}(\alpha)\subset T^*M^{1,0}\otimes\beta\,,\quad A^{\varphi^{\perp}}_{(1,0)}(\beta)\subset T^*M^{1,0}\otimes\alpha\,.$

Then, $\tilde{\varphi}$ is also a pluriharmonic map from M into $G_t(\boldsymbol{C}^n)$ for some t.

REMARK. We may use $A^{\varphi}_{(0,1)}$ and $A^{\varphi^{\perp}}_{(0,1)}$ in place of $A^{\varphi}_{(1,0)}$ and $A^{\varphi^{\perp}}_{(1,0)}$, respectively. In this case, α and β are chosen to be anti-holomorphic subbundles of $V(\varphi)$ and $V(\varphi^{\perp})$, respectively.

To give the examples of α and β which satisfy the conditions (1), (2) of Proposition 2.1, we consider $A^{\varphi}_{(1,0)}$ as a bundle homomorphism $A^{\varphi}_{(1,0)}: TM^{1,0}\otimes V(\varphi)\to V(\varphi^{\perp})$ and set $\mathrm{Im}\,A^{\varphi}_{(1,0)}=\bigcup_{x\in M}\mathrm{Im}\,(A^{\varphi}_{(1,0)})_x$. $\mathrm{Im}\,A^{\varphi}_{(1,0)}$ is a holomorphic subbundle of $V(\varphi^{\perp})$ over $M\setminus W$, where W is an analytic subset of M. It can be observed that $\mathrm{Im}\,A^{\varphi}_{(1,0)}$ extends to a holomorphic subbundle, denoted by $\underline{\mathrm{Im}}\, A^{\varphi}_{(1,0)}$, of $V(\varphi^{\perp})$ over $M\setminus S$, where S is an analytic subset of codimension at least 2. Similarly, considering $A^{\varphi}_{(1,0)}$ as another homomorphism $A^{\varphi}_{(1,0)}: V(\varphi)\to T^{*}M^{1,0}\otimes V(\varphi^{\perp})$ we set $\mathrm{Ker}\, A^{\varphi}_{(1,0)}=\bigcup_{x\in M}\mathrm{Ker}(A^{\varphi}_{(1,0)})_x$. In the same way as above, $\mathrm{Ker}\, A^{\varphi}_{(1,0)}$ extends to a holomorphic subbundle, denoted by $\underline{\mathrm{Ker}}\, A^{\varphi}_{(1,0)}$, of $V(\varphi)$ over $M\setminus S'$, where S' is an analytic subset of codimension at least 2. When we construct a new pluriharmonic map from an old one, we have a new singularity set, hence we give the following definition:

DEFINITION. Denote by S_{φ} the singularity set of M with $\mathrm{codim}_{\boldsymbol{C}} S_{\varphi}\geq 2$ such that φ is a pluriharmonic map from $M\setminus S_{\varphi}$. S_{φ} is of the form $S_{\varphi}=\bigcup_{j=1}^{k} S_j$ for some positive integer k and each S_i $(i=1,\dots,k)$ is an analytic subset of $M\setminus\bigcup_{j=1}^{i-1} S_j$ with $\mathrm{codim}_{\boldsymbol{C}} S_i\geq 2$.

The following lemma enables us to use the method of Burstall-Wood even if the singularity set appears:

LEMMA 2.1 (cf. [O-U2]). *Assume that M is a compact complex manifold with positive first Chern class $c_1(M)>0$. Let E be a Hermitian holomorphic vector bundle over $M\setminus S$, where S is as in the Definition with or without the assumption on φ, and let A be a holomorphic multi-differential with values in* End(E). *Then, A is nilpotent, that is, $A^{m}\equiv 0$ as a holomorphic multi-differential with values in* End(E) *for some positive integer $m\leq \mathrm{rank}\, E$.*

For example, $A^{\varphi^{\perp}}_{(1,0)}\circ A^{\varphi}_{(1,0)}$ is a holomorphic quadratic differential with values in $\mathrm{End}(V(\varphi))$ over $M\setminus S_{\varphi}$, hence nilpotent by Lemma 2.1 if M is compact and $c_1(M)>0$. In particular, $A^{\varphi^{\perp}}_{(1,0)}\circ A^{\varphi}_{(1,0)}$ has the non-trivial kernel. In this case, any non-zero holomorphic subbundle α of $V(\varphi)$ contained in $\mathrm{Ker}(A^{\varphi^{\perp}}_{(1,0)}\circ A^{\varphi}_{(1,0)})$ satisfies the conditions (1) and (2) of Proposition 2.1 with $\beta=\underline{\mathrm{Im}}(A^{\varphi}_{(1,0)}|_{\alpha})$ (see Lemma 2.2 below for the holomorphy of $A^{\varphi}_{(1,0)}|_{\alpha}$).

LEMMA 2.2 (cf. [B-W], [O-U2]). *Let τ and μ be Hermitian vector bundles over M with the Koszul-Malgrange holomorphic structures and let A be a holomorphic multi-differential with values in* $\mathrm{Hom}(\tau,\mu)$. *Then, the following statements are true*:

(1) *If α is a holomorphic subbundle of τ, then $A|_{\alpha}$ is holomorphic.*

(2) *If β is an anti-holomorphic subbundle of μ and $\pi:\mu\to\beta$ is a Hermitian orthogonal projection, then $\pi\circ A$ is holomorphic.*

(3) *If γ is a subbundle of τ with $\tau\ominus\gamma\subset \mathrm{Ker}\, A$ and if γ has the Koszul-Malgrange holomorphic structure with respect to the connection induced from τ, then $A|_{\gamma}$ is a holomorphic multi-differential with values in* $\mathrm{Hom}(\gamma,\mu)$.

(4) *If δ is a subbundle of μ containing the image of A and if δ has the Koszul-Malgrange holomorphic structure with respect to the connection induced from μ, then A is a holomorphic multi-differential with values in* $\mathrm{Hom}(\tau,\delta)$.

In summary, we state the following:

PROPOSITION 2.2. *Let $\varphi: M \setminus S_\varphi \to G_k(\boldsymbol{C}^n)$ be a pluriharmonic map. Then, the following map $\tilde{\varphi}$ defines a pluriharmonic map $M \setminus S_{\tilde{\varphi}} \to G_t(\boldsymbol{C}^n)$ for some t:*

(2.2) $V(\tilde{\varphi}) = \underline{\mathrm{Im}}\, A^\varphi_{(1,0)}$ *if* $A^\varphi_{(1,0)} \not\equiv 0$.

(2.3) $V(\tilde{\varphi}) = V(\varphi) \ominus \underline{\mathrm{Ker}}\, A^\varphi_{(1,0)}$ *if* $\underline{\mathrm{Ker}}\, A^\varphi_{(1,0)} \neq 0$.

(2.4) $V(\tilde{\varphi}) = (V(\varphi) \ominus \alpha) \oplus \underline{\mathrm{Im}}(A^\varphi_{(1,0)}|_\alpha)$, *where α is a holomorphic subbundle of $V(\varphi)$ contained in* $\mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^\varphi_{(1,0)})$, *if $\alpha \neq 0$, which is satisfied if M is compact and $c_1(M) > 0$.*

However, (2.3) may be considered as a special case of (2.4) because $\underline{\mathrm{Ker}}\, A^\varphi_{(1,0)}$ is contained in $\mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^\varphi_{(1,0)})$. Moreover, if M is compact and $c_1(M) > 0$, then (2.2) is also obtained by successive application of the procedure of type (2.4), which follows from more general Proposition 2.3 below. For notational simplicity, we give:

DEFINITION (cf. [B-W]). Set $G^{(1)}(\varphi) = G'(\varphi) = \underline{\mathrm{Im}}\, A^\varphi_{(1,0)}$ and inductively define the *r-th ∂'-Gauss bundle* $G^{(r)}(\varphi)$ of φ by

$$G^{(i+1)}(\varphi) = G'(G^{(i)}(\varphi)) \qquad \text{for} \quad i = 1, 2, \dots.$$

Similarly, define the *r-th ∂''-Gauss bundle* $G^{(-r)}(\varphi)$ by

$$G^{(-1)}(\varphi) = G''(\varphi) = \underline{\mathrm{Im}}\, A^\varphi_{(0,1)}, \quad G^{(-i-1)}(\varphi) = G''(G^{(-i)}(\varphi)) \qquad \text{for} \quad i = 1, 2, \dots.$$

In particular, set $G'_\varphi(\alpha) = \underline{\mathrm{Im}}(A^\varphi_{(1,0)}|_\alpha)$ and $G''_\varphi(\gamma) = \underline{\mathrm{Im}}(A^\varphi_{(0,1)}|_\gamma)$ for a holomorphic subbundle α of $V(\varphi)$ and an anti-holomorphic subbundle γ of $V(\varphi)$, respectively.

PROPOSITION 2.3. *Assume that M is compact and $c_1(M) > 0$. Let $\varphi: M \setminus S_\varphi \to G_k(\boldsymbol{C}^n)$ be a pluriharmonic map and define $\tilde{\varphi}$ by $V(\tilde{\varphi}) = (V(\varphi) \ominus \alpha) \oplus \beta$, where α and β satisfy the conditions* (1) *and* (2) *of Proposition* 2.1. *Then, there is a finite sequence $\{\varphi_i\}_{i=0}^N$ of pluriharmonic maps such that* (1) $\varphi = \varphi_0$, (2) $\tilde{\varphi} = \varphi_N$, (3) *for $i = 0, 1, \dots, N-2$, each φ_{i+1} is obtained from φ_i by $V(\varphi_{i+1}) = (V(\varphi_i) \ominus \alpha_i) \oplus G'_{\varphi_i}(\alpha_i)$, where α_i is a holomorphic subbundle of $V(\varphi_i)$ contained in* $\mathrm{Ker}(A^{\varphi_i^\perp}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$, *and φ_N is obtained by either of the following*:

(I) *If $\beta = G'_\varphi(\alpha)$, φ_N is also obtained from φ_{N-1} by the procedure* (3) *for $i = N-1$,*

(II) *If $\beta \neq G'_\varphi(\alpha)$, there is a holomorphic subbundle β_{N-1} of $(V(\varphi_{N-1}) \oplus G''(\varphi_{N-1}))^\perp$ so that φ_N is obtained from φ_{N-1} by $V(\varphi_N) = V(\varphi_{N-1}) \oplus \beta_{N-1}$.*

PROOF. By Lemma 2.2, $A^{\varphi^\perp}_{(1,0)} \circ A^\varphi_{(1,0)}|_\alpha$ is a holomorphic quadratic differential with values in $\mathrm{End}(\alpha)$. It follows from Lemma 2.1 that $A^{\varphi^\perp}_{(1,0)} \circ A^\varphi_{(1,0)}|_\alpha$ is nilpotent. The rest of the proof proceeds in the same way as the one for Proposition 2.12 in [Wd1]. q.e.d.

DEFINITION (cf. [Wd1]). We call the procedure (2.4) the *forward replacement* of φ by α, and call the procedure like (II) in Proposition 2.3 the *forward extension* of φ_{N-1} by β_{N-1}. When, we use the (0, 1)-part of the second fundamental form and an anti-holomorphic subbundle, we call the corresponding procedures the *backward replacement* and *backward extension*.

To better understand Proposition 2.3, a certain diagram called Salamon's diagram in [B-W] is useful.

Let h_0 be the standard Hermitian fibre metric on $V(\boldsymbol{C}^n)=M\times\boldsymbol{C}^n$. Let $\tau_1,\dots,\tau_k$ be a set of mutually orthogonal subbundles of $V(\boldsymbol{C}^n)$ with respect to h_0 such that each τ_i $(i=1,\dots,k)$ has the Koszul-Malgrange holomorphic structure compatible with the Hermitian structure induced from h_0 and $V(\boldsymbol{C}^n)=\bigoplus_{j=1}^k\tau_j$. Denote by $A^{\tau_i,\tau_j}_{(1,0)}$ the (1, 0)-*second fundamental form* of τ_i in $\tau_i\oplus\tau_j$ for $1\le i\ne j\le k$.

DEFINITION (cf. [B-W]). By a *diagram* $\{\tau_i, A^{\tau_i,\tau_j}_{(1,0)}\}$ we mean the directed graph with vertices $\tau_1,\dots,\tau_k$ and for each pair (i,j) and edge from τ_i to τ_j representing $A^{\tau_i,\tau_j}_{(1,0)}$. The absence of an edge in the graph indicates the vanishing of the corresponding (1, 0)-second fundamental form.

Some statements of Lemma 2.2 are expressed by this diagram as follows:

LEMMA 2.3 (cf. [B-W]). *Given a diagram* $\{\tau_i, A^{\tau_i,\tau_j}_{(1,0)}\}$, $A^{\tau_i,\tau_j}_{(1,0)}:\tau_i\otimes TM^{1,0}\to\tau_j$ *is holomorphic if the diagram contains no configurations of the forms in Figure* 1.

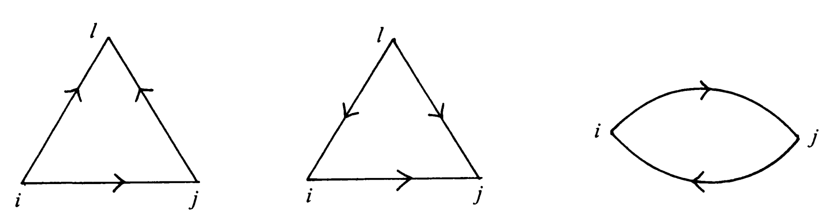
If $A^{\tau_i,\tau_{i+1}}_{(1,0)}(1\le i\le k-1)$ and $A^{\tau_k,\tau_1}_{(1,0)}$ are all holomorphic, we see that the composite $A^{\tau_k,\tau_1}_{(1,0)}\circ A^{\tau_{k-1},\tau_k}_{(1,0)}\circ\cdots\circ A^{\tau_1,\tau_2}_{(1,0)}$ is a holomorphic section of $\bigotimes^k T^*M^{1,0}\otimes\mathrm{End}(\tau_1)$ by Leibniz' rule, hence nilpotent. We often refer to it as a *holomorphic circuit* and denote it by $\{\tau_1,\tau_2,\dots,\tau_k,\tau_1\}$ for notational simplicity.

Let $\varphi: M\to G_k(\boldsymbol{C}^n)$ be a pluriharmonic map from a complex manifold.

DEFINITION (cf. [BD-W1, 2], [B-W]). We say that φ has *∂'-isotropy order r* if $V(\varphi)$ is orthogonal to each $G^{(i)}(\varphi)$ $(1\le i\le r)$ and not orthogonal to $G^{(r+1)}(\varphi)$ with respect to h_0. Moreover, we say that φ has *finite* (resp. *infinite*) ∂'-isotropy order if $r<\infty$ (resp. $r=\infty$). Similarly, the corresponding notion of *∂''-isotropy order* for the ∂''-Gauss bundle is defined.

Note that $V(\varphi)\perp G'(\varphi)$ and $V(\varphi)\perp G''(\varphi)$ always hold, so that any φ has ∂'- and ∂''-isotropy order ≥ 1.

LEMMA 2.4 (cf. [O-U2]). *If φ has ∂'-isotropy order $\ge r$, then $G^{(i)}(\varphi)\perp G^{(j)}(\varphi)$ for any i, j such that $0<|i-j|\le r$.*



(1) (2) (3)

where $1\le l\le k$ with $l\ne i, j$.

FIGURE 1.

If φ has ∂'-isotropy order $\geq r$, then, by Lemma 2.4 we may set

$$R=V(\varphi^{\perp})\ominus\left(\bigoplus_{j=1}^{r} G^{(j)}(\varphi)\right).$$

It follows from Proposition 1.2 and Lemma 2.2 (3) that $V(\varphi)$, $G^{(i)}(\varphi)$ $(1\leq i\leq r)$ and R all have the Koszul-Malgrange holomorphic structures compatible with the Hermitian structures induced from h_0, and $A_{(1,0)}^{V(\varphi),G'(\varphi)}$ and $A_{(1,0)}^{G^{(i)}(\varphi),G^{(i+1)}(\varphi)}$ $(1\leq i\leq r-1)$ are all holomorphic. We often use this fact, without any comment, in the sequel.

If φ is a holomorphic map, then $A^{\varphi}_{(0,1)}=-(A^{\varphi^{\perp}}_{(1,0)})^{*}\equiv 0$, so that $A^{\varphi^{\perp}}_{(1,0)}\equiv 0$ and $V(\varphi)\perp G^{(i)}(\varphi)$ for any $i\geq 1$. Therefore, *a holomorphic map has infinite ∂'-isotropy order.* In the same way, we see that *an anti-holomorphic map has infinite ∂''-isotropy order.*

Given a pluriharmonic map φ of infinite ∂'-isotropy order, we see by Lemma 2.4 that there is a positive integer s such that $G^{(s)}(\varphi)=0$. Therefore, $G^{(s-1)}(\varphi)$ defines an anti-holomorphic map. When the target manifold is a complex projective space CP^{n-1} with the Fubini-Study metric, it turns out that any pluriharmonic map from a compact complex manifold M with $c_1(M)>0$ has infinite ∂'- and ∂''-isotropy order:

THEOREM 2.1 (cf. [O-U2]). *Assume that M is compact and $c_1(M)>0$. Let $\varphi: M\setminus S_{\varphi}\to CP^{n-1}$ be a pluriharmonic map. Then, $G^{(s)}(\varphi)=0$ for some positive integer $s\leq n-1$. Moreover, if φ is non-holomorphic, each $G^{(i)}(\varphi)$ $(0\leq i\leq s-1)$ defines a pluriharmonic map into CP^{n-1}, and $G^{(s-1)}(\varphi)$ defines an anti-holomorphic map.*

3. Pluriharmonic maps into $G_2(C^n)$. In this section, we give a method of constructing a pluriharmonic map $\varphi: M\setminus S_{\varphi}\to G_2(C^n)$, where M is a compact complex manifold with $c_1(M)>0$. We may assume that φ has finite ∂'-isotropy order.

Let r (≥ 1) be the ∂'-isotropy order of a pluriharmonic map $\varphi: M\setminus S_{\varphi}\to G_2(C^n)$. Set

$$A_{r,\varphi}=A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}\circ A_{(1,0)}^{G^{(r-1)}(\varphi),G^{(r)}(\varphi)}\circ\cdots\circ A_{(1,0)}^{V(\varphi),G'(\varphi)}.$$

LEMMA 3.1. *$A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}$ is holomorphic and $A^2_{r,\varphi}\equiv 0$. Define φ_1 from φ by the forward replacement of α^0_0, where $\alpha^0_0=\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}\subset\mathrm{Ker}\,A_{r,\varphi}$ and $\mathrm{rank}\,\alpha^0_0=\mathrm{rank}\,V(\varphi)-1$. Then, either φ_1 is a pluriharmonic map into CP^{n-1} or, φ_1 has ∂'-isotropy order $r+1$ and has the following properties*:

(1) *$A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}$ is holomorphic and $A^2_{r+1,\varphi_1}\equiv 0$.*

(2) *Set $\alpha^1_0=\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}\subset\mathrm{Ker}\,A_{r+1,\varphi_1}$ and set $\alpha^0_1=G'_{\varphi}(\alpha^0_0)$. Then, $\mathrm{rank}\,\alpha^1_0=\mathrm{rank}\,V(\varphi_1)-1$ and the Hermitian orthogonal projection $P_1:\alpha^1_0\to\alpha^0_1$ is a holomorphic isomorphism.*

(3) *Set $R=V(\varphi^{\perp})\ominus(\bigoplus_{j=1}^{r}G^{(j)}(\varphi))$ and set $\alpha^0_r=G^{(r)}_{\varphi}(\alpha^0_0)$. Then, $A_{(1,0)}^{\alpha^0_r,R}$ is holomorphic. Set $R'_0=R\ominus\underline{\mathrm{Im}}\,A_{(1,0)}^{\alpha^0_r,R}$. Then, $G^{(r+1)}(\varphi_1)\subset R'_0\oplus\alpha^0_0$.*

(4) *Set $\alpha^1_j=G^{(j)}_{\varphi_1}(\alpha^1_0)$ for $j=1,\ldots,r+2$. Then, $\alpha^1_{r+1}\subset R'_0$ and $\alpha^1_{r+2}\subset R'_0\oplus\alpha^0_0$.*

(5) *Set $R'_1=((R'_0\oplus\alpha^0_0)\ominus G^{(r+1)}(\varphi_1))\ominus\alpha^1_{r+2}$. Then, $R'_1\perp\alpha^0_1$ and $A_{(1,0)}^{R'_1,\alpha^1_1}\equiv 0$.*

PROOF. By Lemma 2.4 we have a diagram as in Figure 2, where $R=V(\varphi^{\perp})\ominus(\bigoplus_{j=1}^{r}G^{(r)}(\varphi))$. First, we show that $A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}$ is holomorphic. If $r=1$, then $G^{(r)}(\varphi)$ is a holomorphic subbundle of $V(\varphi^{\perp})$ and $A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}=A_{(1,0)}^{\varphi^{\perp}}|_{G^{(r)}(\varphi)}$ is holomorphic by Proposition 1.1 and Lemma 2.2. If $r\geq 2$, then, by Lemma 2.3, $A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}$ is holomorphic. Therefore, $A_{r,\varphi}$ is a holomorphic differential with values in $\mathrm{End}(V(\varphi))$. Then, by Lemma 2.1 we have $A_{r,\varphi}^{2}\equiv 0$, so that $\alpha_0^0=\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}\subset\mathrm{Ker}\,A_{r,\varphi}\subset\mathrm{Ker}(A_{(1,0)}^{\varphi^{\perp}}\circ A_{(1,0)}^{\varphi})$ and $\mathrm{rank}\,\alpha_0^0=1$. Set $\alpha_i^0=G_{\varphi}^{(i)}(\alpha_0^0)$ for $i=1,\dots,r$, and set $\gamma_0^0=V(\varphi)\ominus\alpha_0^0$, $\gamma_i^0=G^{(i)}(\varphi)\ominus\alpha_i^0$ for $i=1,\dots,r$. Then, we have a diagram as in Figure 3. By Lemma 2.3, we see that $A_{(1,0)}^{\alpha_i^0,\alpha_{i+1}^0}$, $A_{(1,0)}^{\gamma_i^0,\gamma_{i+1}^0}$ $(0\leq i\leq r-1)$, $A_{(1,0)}^{\gamma_r^0,\alpha_0^0}$ and $A_{(1,0)}^{\alpha_r^0,R}$ are all holomorphic. Further, set $\alpha_{r+1}^0=\underline{\mathrm{Im}}\,A_{(1,0)}^{\alpha_r^0,R}$ and set $R_0'=R\ominus\alpha_{r+1}^0$. Again, we have a diagram as in Figure 4. By Figure 4 and Lemma 2.3, we see that $A_{(1,0)}^{\alpha_{r+1}^0,\gamma_0^0}$ is also holomorphic. We have a holomorphic circuit $\{\alpha_0^0,\alpha_1^0,\dots,\alpha_{r+1}^0,\gamma_0^0,\gamma_1^0,\dots,\gamma_r^0,\alpha_0^0\}$, which must vanish by Lemma 2.1. However, since $A_{(1,0)}^{\gamma_i^0,\gamma_{i+1}^0}$ $(0\leq i\leq r-1)$ and $A_{(1,0)}^{\gamma_r^0,\alpha_0^0}$ are all surjective and $\mathrm{rank}\,\gamma_0^0=1$, we obtain $A_{(1,0)}^{\alpha_{r+1}^0,\gamma_0^0}\equiv 0$. Hereafter, if $\alpha_i^0=0$ for some $1\leq i\leq r+1$ we understand that $A_{(1,0)}^{\alpha_{r+1}^0,\gamma_0^0}\equiv 0$ is trivially satisfied. Set $V(\varphi_1)=(V(\varphi)\ominus\alpha_0^0)\oplus\alpha_1^0$. If $\alpha_1^0=0$, then $\mathrm{rank}\,V(\varphi_1)=1$ and φ_1 is a

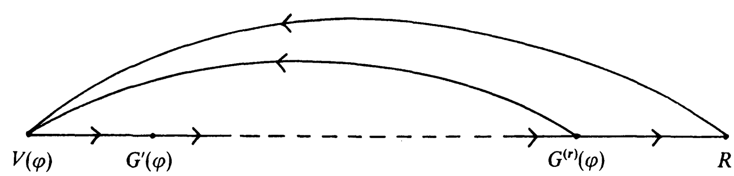


FIGURE 2.

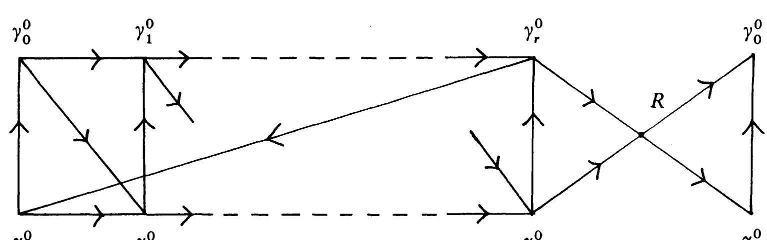


FIGURE 3.

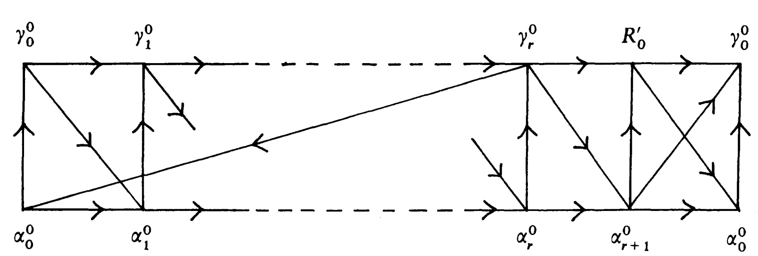


FIGURE 4.

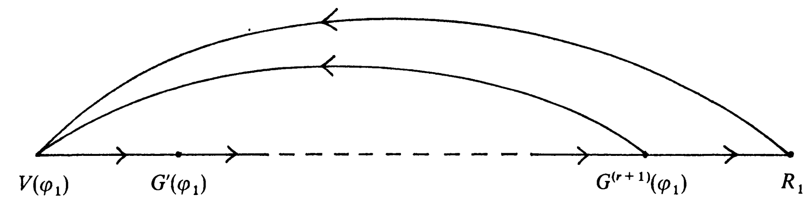


FIGURE 5.

pluriharmonic map into CP^{n-1}. Therefore, we assume that $\alpha_1^0 \neq 0$. Then, by Figure 4 we have

$$V(\varphi_1)=\gamma_0^0 \oplus \alpha_1^0\,, \quad G^{(i)}(\varphi_1)=\gamma_i^0 \oplus \alpha_{i+1}^0 (1 \leq i \leq r)\,, \quad G^{(r+1)}(\varphi_1) \subset R_0' \oplus \alpha_0^0\,,$$

so that φ_1 has ∂'-isotropy order $r+1$. Further, we investigate the properties of φ_1. Setting $R_1=(R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)$, we have a diagram as in Figure 5. By Figure 5 and Lemma 2.3, we see that $A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}$ is holomorphic. We have a holomorphic circuit $\{V(\varphi_1), G'(\varphi_1), \dots, G^{(r+1)}(\varphi_1), V(\varphi_1)\}$. Setting

$$A_{r+1,\varphi_1}=A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)} \circ A_{(1,0)}^{G^{(r)}(\varphi_1),G^{(r+1)}(\varphi_1)} \circ \cdots \circ A_{(1,0)}^{V(\varphi_1),G'(\varphi_1)}$$

we see that A_{r+1,φ_1} is nilpotent. Set $\alpha_0^1=\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}$. Then, $\operatorname{rank}\alpha_0^1 \leq \operatorname{rank} V(\varphi_1) -1$. Let $\tilde{P}^1 : G^{(r+1)}(\varphi_1) \to \alpha_0^0$ and $P_1 : \alpha_0^1 \to \alpha_1^0$ be the Hermitian orthogonal projections. It follows from the surjectivity of $A_{(1,0)}^{\gamma_r^0,\alpha_0^0}$ and the fact that $G^{(r)}(\varphi_1)=\gamma_r^0 \oplus \alpha_{r+1}^0$ that $\tilde{P}^1$ is surjective. Since $(R_0' \oplus \alpha_0^0) \perp \alpha_1^0$, $G^{(r+1)}(\varphi_1) \subset R_0' \oplus \alpha_0^0$ and $A_{(1,0)}^{R_0',\alpha_1^0} \equiv 0$ by Figure 4, we obtain

$$P_1 \circ A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}(v)=A_{(1,0)}^{G^{(r+1)}(\varphi_1),\alpha_1^0}(v)=A_{(1,0)}^{\alpha_0^0,\alpha_1^0} \circ \tilde{P}^1(v)\,, \tag{3.1}$$

where $v \in C^\infty(G^{(r+1)}(\varphi_1))$, which, together with the surjectivity of $\tilde{P}^1$ and $A_{(1,0)}^{\alpha_0^0,\alpha_1^0}$, implies that P_1 is surjective. There, we have $\operatorname{rank}\alpha_0^1 \geq \operatorname{rank}\alpha_1^0=\operatorname{rank} V(\varphi_1)-1$, which, together with the opposite inequality above, implies that $\operatorname{rank}\alpha_0^1=\operatorname{rank}\alpha_1^0=\operatorname{rank} V(\varphi_1)-1$ and P_1 is an isomorphism. P_1 is holomorphic by Lemma 2.2. Now, we show that $A_{r+1,\varphi_1}^2 \equiv 0$. Set $\alpha_i^1=G_{\varphi_1}^{(i)}(\alpha_0^1)$, which is a holomorphic subbundle of $G^{(i)}(\varphi_1)$, for $i=1,\dots,r+1$. If $\tilde{P}^1|_{\alpha_{r+1}^1} : \alpha_{r+1}^1 \to \alpha_0^0$ is surjective, it follows from (3.1) that $P_1(\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}|_{\alpha_{r+1}^1}))=P_1(\alpha_0^1)$, hence $\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}|_{\alpha_{r+1}^1})=\alpha_0^1$, which contradicts the nilpotency of A_{r+1,φ_1}. Therefore, $\tilde{P}^1|_{\alpha_{r+1}^1} \equiv 0$ by $\operatorname{rank}\alpha_0^0=1$, and hence $\alpha_{r+1}^1 \subset \operatorname{Ker} A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}$ by (3.1) and the isomorphy of P_1. Thus, we have proved that $A_{r+1,\varphi_1}^2 \equiv 0$. Moreover, we obtain $\alpha_{r+1}^1 \subset R_0'$ and $\alpha_{r+2}^1=\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1)}|_{\alpha_{r+1}^1}) \subset R_1 \subset R_0' \oplus \alpha_0^0$. Finally, set

$$R_1'=R_1 \ominus \alpha_{r+2}^1=((R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \alpha_{r+2}^1\,.$$

Then, by Figure 4 we see that $R_1' \perp \alpha_1^0$ and $A_{(1,0)}^{R_1',\alpha_1^1} \equiv 0$. q.e.d.

REMARK. When $\dim_{\boldsymbol{C}} M=1$, $A_{r+1,\varphi_1}^2 \equiv 0$ is trivially satisfied. However, when $\dim_{\boldsymbol{C}} M \geq 2$, $A_{r+1,\varphi_1}^2 \equiv 0$ is far from trivial since there is no assurance that $\operatorname{rank} V(\varphi_1)=2$. When we prove $A_{r+1,\varphi_1}^2 \equiv 0$, we fully used the fact that $\operatorname{rank}\,\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r)}(\varphi),V(\varphi)}=1$. On the other hand, there is no longer assurance that rank $\underline{\mathrm{Im}}\, A_{(1,0)}^{G^{(r+1)}(\varphi_1),V(\varphi_1)}=1$. Therefore, when we repeat this procedure, we again encounter difficulty at the next step. However, we can prove the following:

PROPOSITION 3.1. *Set $\varphi_0=\varphi$. For $i=0,1,\dots,$ if φ_i has ∂'-isotropy order $r+i$, $A_{(1,0)}^{G^{(r+1)}(\varphi_i),V(\varphi_i)}$ is holomorphic and $A_{r+i,\varphi_i}^2\equiv 0$, then define φ_{i+1} from φ_i by the forward replacement of α_0^i, where $\alpha_0^i=\underline{\mathrm{Im}}\, A_{(1,0)}^{G^{(r+i)}(\varphi_i),V(\varphi_i)}\subset \mathrm{Ker}\, A_{r+i,\varphi_i}$ and* rank $\alpha_0^i=$ rank $V(\varphi_i)-1$. *Then, either φ_{i+1} is a pluriharmonic map into $\boldsymbol{C}P^{n-1}$ or, φ_{i+1} has ∂'-isotropy order $r+i+1$ and has the following properties*:

(1) *$A_{(1,0)}^{G^{(r+i+1)}(\varphi_{i+1}),V(\varphi_{i+1})}$ is holomorphic and $A_{r+i+1,\varphi_{i+1}}^2\equiv 0$.*

(2) *Set $\alpha_0^{i+1}=\underline{\mathrm{Im}}\, A_{(1,0)}^{G^{(r+i+1)}(\varphi_{i+1}),V(\varphi_{i+1})}\subset \mathrm{Ker}\, A_{r+i+1,\varphi_{i+1}}$ and set $\alpha_1^i=G'_{\varphi_i}(\alpha_0^i)$. Then,* rank $\alpha_0^{i+1}=$ rank $V(\varphi_{i+1})-1$ *and the Hermitian orthogonal projection $P_{i+1}\colon \alpha_0^{i+1}\to\alpha_1^i$ is a holomorphic isomorphism.*

(3) *$G^{(r+s)}(\varphi_{i+1})\subset R'_{s-1}\oplus\alpha_0^{s-1}$ $(1\le s\le i+1)$.*

(4) *Set $\alpha_j^{i+1}=G_{\varphi_{i+1}}^{(j)}(\alpha_0^{i+1})$ for $j=1,\dots,r+i+2$. Then, $\alpha_{r+s}^{i+1}\subset R'_{s-1}$ $(1\le s\le i+1)$ and $\alpha_{r+i+2}^{i+1}\subset R'_i\oplus\alpha_0^i$.*

(5) *Set $R'_{i+1}=((R'_i\oplus\alpha_0^i)\ominus G^{(r+i+1)}(\varphi_{i+1}))\ominus\alpha_{r+i+2}^{i+1}$. Then, $R'_{i+1}\perp\alpha_1^i$ and $A_{(1,0)}^{R'_{i+1},\alpha_1^{i+1}}\equiv 0$.*

PROOF. For $i=0$, Proposition 3.1 holds by Lemma 3.1. Assume that Proposition 3.1 is true for $0\le i\le k$ and φ_{i+1} $(0\le i\le k)$ is not a map into $\boldsymbol{C}P^{n-1}$, so that each φ_{i+1} $(0\le i\le k)$ has the properties (1)–(5). Then, we may define φ_{k+2} from φ_{k+1} by the forward replacement of α_0^{k+1}. If $\alpha_1^{k+1}=0$, then rank $V(\varphi_{k+2})=1$ by (2) for φ_{k+1}, and φ_{k+2} is a pluriharmonic map into $\boldsymbol{C}P^{n-1}$. Hence, we may assume that $\alpha_1^{k+1}\neq 0$. Now, we draw the diagram for φ_{i+1} $(0\le i\le k)$. Set $\gamma_0^{i+1}=V(\varphi_{i+1})\ominus\alpha_0^{i+1}$ and $\gamma_j^{i+1}=G^{(j)}(\varphi_{i+1})\ominus\alpha_j^{i+1}$ for $j=1,\dots,r+i+1$. By the properties (1)–(5) for φ_{i+1}, we have a diagram as in Figure 6. In particular, when $i=k$, we have a holomorphic circuit

$$\{\alpha_0^{k+1},\alpha_1^{k+1},\dots,\alpha_{r+k+2}^{k+1},\gamma_0^{k+1},\gamma_1^{k+1},\dots,\gamma_{r+k+1}^{k+1},\alpha_0^{k+1}\},$$

which is nilpotent. Since rank $\gamma_0^{k+1}=1$ by (2) for φ_{k+1}, and $A_{(1,0)}^{\gamma_j^{k+1},\gamma_{j+1}^{k+1}}$ $(0\le j\le r+k)$ and $A_{(1,0)}^{\gamma_{r+k+1}^{k+1},\alpha_0^{k+1}}$ are all surjective, we obtain $A_{(1,0)}^{\alpha_{r+k+2}^{k+1},\gamma_0^{k+1}}\equiv 0$. Consequently, it follows from Figure 6 that

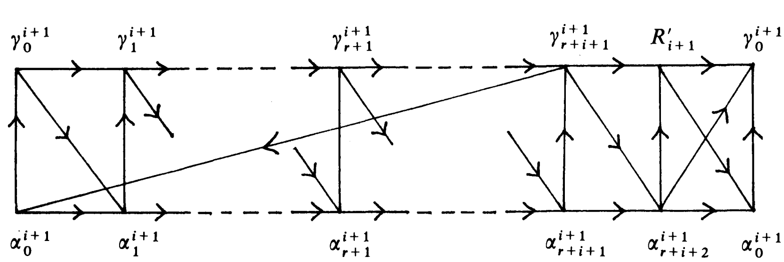


FIGURE 6.

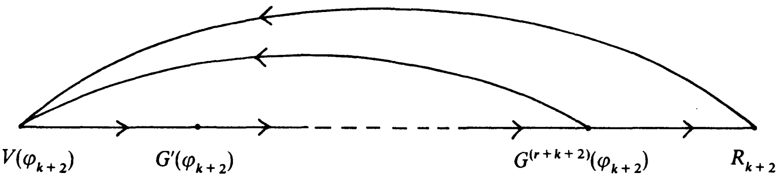


FIGURE 7.

$$(3.2)\qquad V(\varphi_{k+2})=\gamma_0^{k+1}\oplus\alpha_1^{k+1}\,,\quad G^{(j)}(\varphi_{k+2})=\gamma_j^{k+1}\oplus\alpha_{j+1}^{k+1}\qquad(1\le j\le r+k+1)\,,$$
$$G^{(r+k+2)}(\varphi_{k+2})\subset R'_{k+1}\oplus\alpha_0^{k+1}\,.$$

Therefore, φ_{k+2} has ∂'-isotropy order $r+k+2$. By (3) and (4) for φ_{k+1} and the definition of R'_s we obtain

$$\gamma_{r+s}^{k+1}\subset G^{(r+s)}(\varphi_{k+1})\subset R'_{s-1}\oplus\alpha_0^{s-1}\quad(1\le s\le k+1)\,,$$
$$\alpha_{r+s+1}^{k+1}\subset R'_s\subset R'_{s-1}\oplus\alpha_0^{s-1}\quad(1\le s\le k)\,,\quad \alpha_{r+k+2}^{k+1}\subset R'_k\oplus\alpha_0^k\,,$$

which, together with (3.2), yields

$$(3.3)\qquad G^{(r+s)}(\varphi_{k+2})\subset R'_{s-1}\oplus\alpha_0^{s-1}\qquad(1\le s\le k+2)\,.$$

Set $R_{k+2}=V(\varphi_{k+2}^\perp)\ominus(\bigoplus_{j=1}^{r+k+2}G^{(j)}(\varphi_{k+2}))$. We have a diagram as in Figure. 7. By Figure 7 and Lemma 2.3, we see that $A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),V(\varphi_{k+2})}$ is holomorphic. Set $\alpha_0^{k+2}=\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),V(\varphi_{k+2})}$ and $\alpha_j^{k+2}=G_{\varphi_{k+2}}^{(j)}(\alpha_0^{k+2})$ for $j=1,\dots,r+k+2$. Define an $\mathrm{End}(V(\varphi_{k+2}))$-valued holomorphic differential $A_{r+k+2,\varphi_{k+2}}$ as before. Then, this is nilpotent. Therefore, $\operatorname{rank}\alpha_0^{k+2}\le\operatorname{rank}V(\varphi_{k+2})-1$. Let $\tilde{P}^{k+2}\colon G^{(r+k+2)}(\varphi_{k+2})\to\alpha_0^{k+1}$ and $P_{k+2}\colon\alpha_0^{k+2}\to\alpha_1^{k+1}$ be the Hermitian orthogonal projections. It follows from the surjectivity of $A_{(1,0)}^{\gamma_{r+k+1}^{k+1},\alpha_0^{k+1}}$ and the fact that $G^{(r+k+1)}(\varphi_{k+2})=\gamma_{r+k+1}^{k+1}\oplus\alpha_{r+k+2}^{k+1}$ (see Figure 6 and (3.2)) that $\tilde{P}^{k+2}$ is surjective. Since $(R'_{k+1}\oplus\alpha_0^{k+1})\perp\alpha_1^{k+1}$, $G^{(r+k+2)}(\varphi_{k+2})\subset R'_{k+1}\oplus\alpha_0^{k+1}$ and $A_{(1,0)}^{R'_{k+1},\alpha_1^{k+1}}\equiv0$ by Figure 6 for $i=k$, we obtain

$$(3.4)\qquad P_{k+2}\circ A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),V(\varphi_{k+2})}(v)=A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\alpha_1^{k+1}}(v)=A_{(1,0)}^{\alpha_0^{k+1},\alpha_1^{k+1}}\circ\tilde{P}^{k+2}(v)\,,$$

where $v\in C^\infty(G^{(r+k+2)}(\varphi_{k+2}))$, which, together with the surjectivity of $\tilde{P}^{k+2}$ and $A_{(1,0)}^{\alpha_0^{k+1},\alpha_1^{k+1}}$, implies that P_{k+2} is surjective. Therefore, we have

$$\operatorname{rank}\alpha_0^{k+2}\ge\operatorname{rank}\alpha_1^{k+1}=\operatorname{rank}V(\varphi_{k+2})-\operatorname{rank}\gamma_0^{k+1}=\operatorname{rank}V(\varphi_{k+2})-1\,,$$

where the last equality follows from (2) for φ_{k+1}. Consequently, we see that $\operatorname{rank}\alpha_0^{k+2}=\operatorname{rank}\alpha_1^{k+1}=\operatorname{rank}V(\varphi_{k+2})-1$ and P_{k+2} is an isomorphism.

Now, we show that $A_{r+k+2,\varphi_{k+2}}^2\equiv0$. By (3.3) we have

$$(3.5)\qquad \alpha_{r+s}^{k+2}\subset G^{(r+s)}(\varphi_{k+2})\subset R'_{s-1}\oplus\alpha_0^{s-1}\qquad(1\le s\le k+2)\,.$$

First, we must show that $\alpha_{r+s}^{k+2}\subset R'_{s-1}\,(1\le s\le k+2)$. Let $p^s\colon\alpha_{r+s}^{k+2}\to\alpha_0^{s-1}$, $q^s\colon\alpha_{r+s}^{k+2}\to R'_{s-1}\,(1\le s\le k+2)$ and $\tau^s\colon\alpha_{r+s+1}^{k+2}\to\alpha_1^{s-1}\,(1\le s\le k+1)$ be the Hermitian orthogonal projections. Take any $v \in C^\infty(\alpha^{k+2}_{r+s})$. By (3.5), we may set $v = p^s(v) + q^s(v)$. For $1 \le s \le k+1$, we have

$$\tau^s \circ A^{\alpha^{k+2}_{r+s}, \alpha^{k+2}_{r+s+1}}_{(1,0)}(v) = A^{\alpha^{s-1}_0, \alpha^{s-1}_1}_{(1,0)}(p^s(v)) + A^{R'_{s-1}, \alpha^{s-1}_1}_{(1,0)}(q^s(v)) = A^{\alpha^{s-1}_0, \alpha^{s-1}_1}_{(1,0)} \circ p^s(v)\,, \tag{3.6}$$

where we have used the facts $(R'_{s-1} \oplus \alpha^{s-1}_0) \perp \alpha^{s-1}_1$ (see Figure 6) and (5) for φ_{s-1}. If p^s is surjective, then, since $A^{\alpha^{s-1}_0, \alpha^{s-1}_1}_{(1,0)}$ is surjective, (3.6) implies that τ^s is also surjective, where we note that $\alpha^{s-1}_0 \neq 0$ and $\alpha^{s-1}_1 \neq 0$ $(1 \le s \le k+1)$ because neither φ_{s-1} nor φ_s defines a map into CP^{n-1} by assumption. Since $R'_s \perp \alpha^{s-1}_1$, $\operatorname{rank} \alpha^s_0 = \operatorname{rank} \alpha^{s-1}_1$ and $P_s : \alpha^s_0 \to \alpha^{s-1}_1$ is an isomorphism by (2), (5) for φ_s, the surjectivity of τ^s implies that $p^{s+1} : \alpha^{k+2}_{r+s+1} \to \alpha^s_0$ is also surjective. Now, suppose that p^1 is surjective. Then, each p^s $(1 \le s \le k+2)$ is surjective. In particular, $p^{k+2} : \alpha^{k+2}_{r+k+2} \to \alpha^{k+1}_0$ is surjective. Note that $p^{k+2} = \tilde{P}^{k+2}|_{\alpha^{k+2}_{r+k+2}}$. Then, it follows from the surjectivity of p^{k+2} and (3.4) that $P_{k+2}(\underline{\mathrm{Im}}(A^{G^{(r+k+2)}(\varphi_{k+2}), V(\varphi_{k+2})}_{(1,0)}|_{\alpha^{k+2}_{r+k+2}})) = P_{k+2}(\alpha^{k+2}_0)$, so that $\underline{\mathrm{Im}}(A^{G^{(r+k+2)}(\varphi_{k+2}), V(\varphi_{k+2})}_{(1,0)}|_{\alpha^{k+2}_{r+k+2}}) = \alpha^{k+2}_0$, which contradicts the nilpotency of $A_{r+k+2, \varphi_{k+2}}$. Therefore, we have proved that p^1 cannot be surjective, which, together with the fact $\operatorname{rank} \alpha^0_0 = 1$, implies that $p^1 \equiv 0$. For any fixed s $(1 \le s \le k+1)$, if $p^s \equiv 0$, then by (3.6) and the surjectivity of $A^{\alpha^{k+2}_{r+s}, \alpha^{k+2}_{r+s+1}}_{(1,0)}$ for $1 \le s \le k+1$, we see that $\tau^s \equiv 0$, where we note that if $\alpha^{k+2}_{r+s+1} = 0$ then $\tau^s \equiv 0$ is trivially satisfied. Since P_s is an isomorphism, it follows from $\tau^s \equiv 0$ that $p^{s+1} \equiv 0$. Thus, we have proved that $p^s \equiv 0$ $(1 \le s \le k+2)$, which, together with (3.5), yields

$$\alpha^{k+2}_{r+s} \subset R'_{s-1} \qquad (1 \le s \le k+2)\,. \tag{3.7}$$

Moreover, the fact $p^{k+2} \equiv 0$, the isomorphicity of P_{k+2} and (3.4) imply that $\alpha^{k+2}_{r+k+2} \subset \operatorname{Ker} A^{G^{(r+k+2)}(\varphi_{k+2}), V(\varphi_{k+2})}_{(1,0)}$, so that $A^2_{r+k+2, \varphi_{k+2}} \equiv 0$.

Finally, set

$$\alpha^{k+2}_{r+k+3} = \underline{\mathrm{Im}}(A^{G^{(r+k+2)}(\varphi_{k+2})}_{(1,0)}|_{\alpha^{k+2}_{r+k+2}}) \subset R_{k+2} \subset R'_{k+1} \oplus \alpha^{k+1}_0\,,$$

and set

$$R'_{k+2} = R_{k+2} \ominus \alpha^{k+2}_{r+k+3} = ((R'_{k+1} \oplus \alpha^{k+1}_0) \ominus G^{(r+k+2)}(\varphi_{k+2})) \ominus \alpha^{k+2}_{r+k+3}\,.$$

Then, by Figure 6 for $i = k$, we see that $R'_{k+2} \perp \alpha^{k+1}_1$ and $A^{R'_{k+2}, \alpha^{k+2}_1}_{(1,0)} \equiv 0$. q.e.d.

Now, we have the following:

THEOREM 3.1. *Let $\varphi : M \setminus S_\varphi \to G_2(C^n)$ be a pluriharmonic map. Assume that φ has finite ∂'-isotropy order r. Then, there is a sequence $\{\varphi_i\}_{i=0}^N$ of pluriharmonic maps such that*

(1) $\varphi_0 = \varphi$, (2) $\varphi_N : M \setminus S_{\varphi_N} \to CP^{n-1}$, (3) *for $i = 0, 1, \dots, N-1$, each φ_i has ∂'-isotropy order $r+i$, and φ_{i+1} is obtained from φ_i by the forward replacement of $\underline{\mathrm{Im}}\, A^{G^{(r+i)}(\varphi_i), V(\varphi_i)}_{(1,0)}$, which is a holomorphic subbundle of $V(\varphi_i)$ contained in $\operatorname{Ker}(A^{\varphi_i^\perp}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$.*

PROOF. Let N be any positive integer such that each φ_i $(0 \le i \le N)$ is not a map into CP^{n-1}. Then, by Proposition 3.1 we see that φ_N has ∂'-isotropy order $r+N$. However, this is impossible because the ∂'-isotropy order $r+N$ must be less than n.

Therefore, there exists a positive integer N such that φ_N is a pluriharmonic map from $M\setminus S_{\varphi_N}$ into CP^{n-1}, which, together with Proposition 3.1, yields the assertions (1)–(3) of Theorem 3.1. q.e.d.

4. A construction of pluriharmonic maps from rational maps. In this section, we give the procedures inverse to those in Theorems 2.1 and 3.1. For this purpose, we review the following propositions which are higher dimensional versions of Proposition 2.3 and the following Remark and Theorem 4.1 of [B-W], respectively.

PROPOSITION 4.1 (cf. [O-U2]). *Let $\varphi: M\to G_k(C^n)$ be a pluriharmonic map from a complex manifold. Let $\alpha\subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)}\circ A^{\varphi}_{(1,0)})$ be a holomorphic subbundle of $V(\varphi)$ and let $\tilde\varphi$ be defined from φ by the forward replacement of α. Then, $G'_\varphi(\alpha)$ is an anti-holomorphic subbundle of $V(\tilde\varphi)$, $G'_\varphi(\alpha)\subset \mathrm{Ker}(A^{\tilde\varphi^\perp}_{(0,1)}\circ A^{\tilde\varphi}_{(0,1)})$ and, if $\underline{\mathrm{Ker}}\, A^{\varphi}_{(1,0)}=0$, then φ is obtained from $\tilde\varphi$ by the backward replacement of $G'_\varphi(\alpha)$.*

PROPOSITION 4.2 (cf. [O-U2]). *Let $\varphi: M\to G_k(C^n)$ be a pluriharmonic map from a complex manifold. Assume that $\underline{\mathrm{Ker}}\, A^{\varphi}_{(1,0)}\neq 0$. Then, there exists a pluriharmonic map $\psi: M\setminus S_\psi\to G_t(C^n)$ for some $0\le t\le k-1$ and a non-zero anti-holomorphic subbundle β of $(V(\psi)\oplus G'(\psi))^\perp$ such that $V(\varphi)=V(\psi)\oplus\beta$ over $M\setminus S_\psi$. Conversely, given a pluriharmonic map $\psi: M\to G_t(C^n)$ and a non-zero anti-holomorphic subbundle β of $(V(\psi)\oplus G'(\psi))^\perp$ then φ defined by $V(\varphi)=V(\psi)\oplus\beta$ gives a pluriharmonic map φ: $M\setminus S_\varphi\to G_k(C^n)$ with $\underline{\mathrm{Ker}}\, A^{\varphi}_{(1,0)}\neq 0$, where $k=t+\mathrm{rank}\,\beta$.*

Recall that we called the procedure $V(\psi)\to V(\psi)\oplus\beta$ in Proposition 4.2 the backward extension. We remark that if we reverse the orientation of M we may use the concepts of ∂''-isotropy order and the backward replacement in place of those of ∂'-isotropy order and the forward replacement, respectively. For example, given a pluriharmonic map ψ and a non-zero holomorphic subbundle β of $(V(\psi)\oplus G''(\psi))^\perp$, we can produce a new pluriharmonic map φ with $\underline{\mathrm{Ker}}\, A^{\varphi}_{(0,1)}\neq 0$ by $V(\varphi)=V(\psi)\oplus\beta$, and called this procedure the forward extension (cf. Section 2).

First of all, we treat the case of infinite isotropy order.

PROPOSITION 4.3. *Let $\varphi: M\setminus S_\varphi\to G_k(C^n)$ be any non-holomorphic pluriharmonic map with infinite ∂''-isotropy order, where M is a complex manifold. Then, there is a unique sequence $\{\varphi^i\}_{i=0}^N$ of pluriharmonic maps such that*

(1) $\varphi^N=\varphi$, (2) $\varphi^0: M\setminus S_{\varphi^0}\to G_t(C^n)$ is a pluriharmonic map for some $t\in N$, (3) for $i=0,1,\dots,N-1$, $\underline{\mathrm{Ker}}\, A^{\varphi^i}_{(1,0)}=0$, and each φ^{i+1} is obtained from φ^i by $V(\varphi^{i+1})=G'(\varphi^i)\oplus\alpha^i$, where α^i is a holomorphic subbundle of $(G'(\varphi^i)\oplus V(\varphi^i))^\perp$.

RAMARK. (1) Since $G''(G'(\varphi^i))=V(\varphi^i)$ by the condition $\underline{\mathrm{Ker}}\, A^{\varphi^i}_{(1,0)}=0$, the procedure $G'(\varphi^i)\to V(\varphi^{i+1})$ is the forward extension.

(2) A similar result where M is a Riemann surface is already proved in [Wd1].

PROOF. Since $G^{(-s)}(\varphi)=0$ for some $s\in N$, set $V(\varphi^i)=G^{(-s+1+i)}(\varphi)$ for $i=0,1,\dots,$

$s-1$. Since $G'(G^{(-s+1+i)}(\varphi))\subset G^{(-s+2+i)}(\varphi)$, we have $G'(\varphi^i)\subset V(\varphi^{i+1})$. Set $\alpha^i=\underline{\mathrm{Ker}}\,A_{(0,1)}^{\varphi^{i+1},\varphi^i}$, then by (1.2) and Proposition 4.2 we see that $V(\varphi^{i+1})=G'(\varphi^i)\oplus\alpha^i$ and α^i is a holomorphic subbundle of $(G'(\varphi^i)\oplus G''(G'(\varphi^i)))^{\perp}$. Note that the condition $\underline{\mathrm{Ker}}\,A_{(1,0)}^{\varphi^i}=0$ is equivalent to the surjectivity of the map $A_{(0,1)}^{\varphi^{i+1},\varphi^i}: V(\varphi^{i+1})\to V(\varphi^i)$ (cf. (1.2)). Now, $N=s-1$ and the existence is established. For the uniqueness, define the sequence $\{\varphi^i\}_{i=0}^N$ as in (3), where φ^0 is as in (2). We show that each α^i is uniquely determined by the condition (1). Suppose that $V(\varphi^i)\subsetneqq G^{(-N+i)}(\varphi)$ for some $1\le i\le N-1$. Set $\beta^i=G^{(-N+i)}(\varphi)\ominus V(\varphi^i)$. Since, $V(\varphi^{i+1})$ is a holomorphic subbundle of $V(\varphi^i)^{\perp}$, $A_{(0,1)}^{G^{(-N+i+1)}(\varphi),\beta^i}$ is surjective, and $\underline{\mathrm{Ker}}\,A_{(1,0)}^{\varphi^i}=0$, it follows that $V(\varphi^{i+1})$ cannot have $G^{(-N+i+1)}(\varphi)$ as a direct factor and $V(\varphi^{i+1})\subset G^{(-N+i+1)}(\varphi)\oplus\beta^i$. Thus, either $V(\varphi^{i+1})\subsetneqq G^{(-N+i+1)}(\varphi)$ or $V(\varphi^{i+1})$ has a non-trivial projection into β^i. The former case may be treated in the same way, and the latter one yields $\varphi^N\neq\varphi$ because $\underline{\mathrm{Ker}}\,A_{(1,0)}^{\varphi^j}=0$ and $\underline{\mathrm{Ker}}\,A_{(1,0)}^{G^{(-j-1)}(\varphi)}=0$ for any $0\le j\le N-1$. Therefore, we have $\varphi^N\neq\varphi$. Next, suppose that $V(\varphi^i)\supsetneqq G^{(-N+i)}(\varphi)$ for some $1\le i\le N-1$. If $V(\varphi^i)$ contains also $G^{(-N+i+1)}(\varphi)$, then $G^{(-N+i)}(\varphi)\subset \mathrm{Ker}\,A_{(1,0)}^{\varphi^i}$, which is a contradiction. Thus, $V(\varphi^i)$ has a proper holomorphic subbundle of $G^{(-N+i+1)}(\varphi)$ as a direct factor, hence, again, we have $\varphi^N\neq\varphi$. Finally, suppose that $G'(\varphi^{i-1})\subsetneqq G^{(-N+i)}(\varphi)$ and that α^{i-1} has a non-trivial projection into both of $G^{(-N+i+1)}(\varphi)$ and β^i for some $1\le i\le N-1$. This case also leads to the conclusion that $\varphi^N\neq\varphi$. q.e.d.

THEOREM 4.1. *Let $\varphi: M\setminus S_{\varphi}\to \boldsymbol{C}P^{n-1}$ be a pluriharmonic map which is not $\pm$-holomorphic, where M is a compact complex manifold with $c_1(M)>0$. Then, there is a unique sequence $\{\varphi^i\}_{i=0}^N$ $(N\le n-1)$ of pluriharmonic maps into $\boldsymbol{C}P^{n-1}$ such that*

(1) *$\varphi^N=\varphi$,* (2) *$\varphi^0: M\setminus S_{\varphi^0}\to \boldsymbol{C}P^{n-1}$ is a non-constant holomorphic map with* $\mathrm{rank}_{\boldsymbol{C}}\partial\varphi^0\le 1$, *that is, a rational map $f: M\to \boldsymbol{C}P^{n-1}$ with* $\mathrm{rank}_{\boldsymbol{C}}\partial f\le 1$, (3) *for $i=0, 1,\dots, N-1$, each φ^{i+1} is obtained from φ^i by $V(\varphi^{i+1})=G'(\varphi^i)$.*

PROOF. This follows from Theorem 2.1 and Proposition 4.3. q.e.d.

For the case of finite isotropy order, we have the following:

THEOREM 4.2. *Let $\varphi: M\setminus S_{\varphi}\to G_2(\boldsymbol{C}^n)$ be any pluriharmonic map with finite ∂'-isotropy order, where M is a compact complex manifold with $c_1(M)>0$. Then, there is a sequence $\{\varphi^i\}_{i=0}^N$ of pluriharmonic maps such that*

(1) *$\varphi^N=\varphi$,* (2) *$\varphi^0: M\setminus S_{\varphi^0}\to \boldsymbol{C}P^{n-1}$, and φ^1 is obtained from φ^0 by the backward extension of β^0, which is so chosen that $\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r)}(\varphi^1),V(\varphi^1)}=\beta^0$ for some $r\in \boldsymbol{N}$, and that if φ^0 is non-holomorphic then* $\mathrm{rank}\,\beta^0=1$, (3) *for $i=1,\dots, N-1$, each φ^{i+1} has ∂'-isotropy order $r-i$, and φ^{i+1} is obtained from φ^i by the backward replacement of α^i and the backward extension of β^i, where α^i and β^i are so chosen that* $\mathrm{rank}\,\alpha^i=\mathrm{rank}\,V(\varphi^i)-1$ *and the Hermitian orthogonal projection P^i: $\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r+1-i)}(\varphi^i),V(\varphi^i)}\to\alpha^i$ is a holomorphic isomorphism and $\underline{\mathrm{Im}}\,A_{(1,0)}^{G^{(r-i)}(\varphi^{i+1}),V(\varphi^{i+1})}=G''_{\varphi^i}(\alpha^i)\oplus\beta^i$.*

PROOF. This follows from Theorem 3.1 and Propositions 4.1, 4.2. q.e.d.

The uniqueness for the choice of β^i may be expected if we assume that $\underline{\mathrm{Ker}}(A^{\varphi^i}_{(1,0)}|_{\alpha^i})=0$. However, in general, it seems to be difficult to determine α^i uniquely.

5. Examples of pluriharmonic maps with finite ∂'-isotropy order. In this section, we give examples of pluriharmonic maps with ∂'-isotropy order 1 or 2, which are maps of CP^2 into complex Grassmann manifolds.

Let $[\zeta_0, \zeta_1, \zeta_2]$ be a homogeneous coordinate system for CP^2. In an open set U_0 where $\zeta_0 \neq 0$, set $z_1=\zeta_1/\zeta_0$, $z_2=\zeta_2/\zeta_0$.

EXAMPLE 1. Let $\varphi_0: CP^2 \ni [1, z_1, z_2] \to [1, \sqrt{2}z_1, \sqrt{2}z_2, z_1^2, \sqrt{2}z_1z_2, z_2^2] \in CP^5$ be the second Veronese embedding. A line bundle $V(\varphi_0)$ is locally spanned by

$$\xi_0(z_1, z_2)=(1, \sqrt{2}z_1, \sqrt{2}z_2, z_1^2, \sqrt{2}z_1z_2, z_2^2)\,,$$

which extends to a global meromorphic section of $V(\varphi_0)$. Using the expression for ξ_0, we find that $V(\varphi_1)=\underline{\mathrm{Im}}\,A^{\varphi_0}_{(1,0)}$ is locally spanned by ξ_1^1 and ξ_1^2, where

$$\begin{aligned}\xi_1^1=(&-2\bar z_1, \sqrt{2}(1-|z_1|^2+|z_2|^2), -2\sqrt{2}\bar z_1z_2, 2z_1(1+|z_2|^2),\\ &\sqrt{2}z_2(1-|z_1|^2+|z_2|^2), -2\bar z_1z_2^2)\,,\\ \xi_1^2=(&-2\bar z_2, -2\sqrt{2}z_1\bar z_2, \sqrt{2}(1+|z_1|^2-|z_2|^2), -2z_1^2\bar z_2,\\ &\sqrt{2}z_1(1+|z_1|^2-|z_2|^2), 2z_2(1+|z_1|^2))\,.\end{aligned}$$

Again, using the expressions for ξ_1^1 and ξ_1^2, we find that $V(\varphi_2)=\underline{\mathrm{Im}}\,A^{\varphi_1}_{(1,0)}$ is locally spanned by ξ_2^1, ξ_2^2 and ξ_2^3, where

$$\begin{aligned}\xi_2^1&=(2\bar z_1^2, -2\sqrt{2}\bar z_1(1+|z_2|^2), 2\sqrt{2}\bar z_1^2z_2, 2(1+|z_2|^2)^2, -2\sqrt{2}z_2\bar z_1(1+|z_2|^2), 2\bar z_1^2z_2^2)\,,\\ \xi_2^2&=(2\bar z_1\bar z_2, -\sqrt{2}\bar z_2(1-|z_1|^2+|z_2|^2), -\sqrt{2}\bar z_1(1+|z_1|^2-|z_2|^2), -2z_1\bar z_2(1+|z_2|^2),\\ &\quad\sqrt{2}(1+|z_1|^2+|z_2|^2+2|z_1|^2|z_2|^2), -2\bar z_1z_2(1+|z_1|^2))\,,\\ \xi_2^3&=(2\bar z_2^2, 2\sqrt{2}z_1\bar z_2^2, -2\sqrt{2}\bar z_2(1+|z_1|^2), 2z_1^2\bar z_2^2, -2\sqrt{2}z_1\bar z_2(1+|z_1|^2), 2(1+|z_1|^2)^2)\,.\end{aligned}$$

Then, we see that $A^{\varphi_2}_{(1,0)}\equiv 0$, that is, φ_2 is an anti-holomorphic map into $G_3(C^6)$. Since $V(\varphi_2)$ is an anti-holomorphic subbundle of $(V(\varphi_0)\oplus G'(\varphi_0))^\perp$, the map $\psi: CP^2 \to G_4(C^6)$ defined by $V(\psi)=V(\varphi_0)\oplus V(\varphi_2)$ is pluriharmonic by Proposition 4.2. Moreover, ψ has ∂'-isotropy order 1. For the purpose of producing a pluriharmonic map into $G_2(C^6)$ with ∂'-isotropy order 1, we now look for an anti-holomorphic line subbundle of $V(\varphi_2)$.

Let β^0 be spanned by

$$\eta=(0, 0, 0, -\bar z_2^2, \sqrt{2}\bar z_1\bar z_2, -\bar z_1^2)\,.$$

In fact,

$$\eta=-\frac{1}{4}\partial_2\partial_2\xi_2^1=\frac{1}{2}\partial_2\partial_1\xi_2^2=-\frac{1}{4}\partial_1\partial_1\xi_2^3\,.$$

Since $\partial C^\infty(V(\varphi_2))\subset C^\infty(V(\varphi_2))$ and $\partial\eta=0$, we see that β^0 is an anti-holomorphic line subbundle of $V(\varphi_2)$. Define φ^1 by $V(\varphi^1)=V(\varphi_0)\oplus\beta^0$. Then, φ^1 is a pluriharmonic map of $CP^2\setminus S_{\varphi^1}$ into $G_2(C^6)$ with ∂'-isotropy order 1, where S_{φ^1} consists of the isolated point $z_1=z_2=0$.

EXAMPLE 2. Let

$$\varphi_0\colon CP^2\ni[1,z_1,z_2]\to$$
$$(1,\sqrt{3}z_1,\sqrt{3}z_2,\sqrt{3}z_1^2,\sqrt{3}z_2^2,\sqrt{6}z_1z_2,z_1^3,\sqrt{3}z_1^2z_2,\sqrt{3}z_1z_2^2,z_2^3)\in CP^9$$

be the third Veronese embedding. A line bundle $V(\varphi_0)$ is locally spanned by

$$\xi_0(z_1,z_2)=(1,\sqrt{3}z_1,\sqrt{3}z_2,\sqrt{3}z_1^2,\sqrt{3}z_2^2,\sqrt{6}z_1z_2,z_1^3,\sqrt{3}z_1^2z_2,\sqrt{3}z_1z_2^2,z_2^3)\,.$$

$V(\varphi_1)=\underline{\operatorname{Im}}\,A^{\varphi_0}_{(1,0)}$ is locally spanned by ξ_1^1 and ξ_1^2, where

$$\begin{aligned}\xi_1^1=(&-3\bar z_1,\sqrt{3}(1-2|z_1|^2+|z_2|^2),-3\sqrt{3}\bar z_1z_2,\sqrt{3}z_1(2-|z_1|^2+2|z_2|^2),\\&-3\sqrt{3}\bar z_1z_2^2,\sqrt{6}z_2(1-2|z_1|^2+|z_2|^2),3z_1^2(1+|z_2|^2),\\&\sqrt{3}z_1z_2(2-|z_1|^2+2|z_2|^2),\sqrt{3}z_2^2(1-2|z_1|^2+|z_2|^2),-3\bar z_1z_2^3)\,,\\ \xi_1^2=(&-3\bar z_2,-3\sqrt{3}\bar z_2z_1,\sqrt{3}(1+|z_1|^2-2|z_2|^2),-3\sqrt{3}z_1^2\bar z_2,\\&\sqrt{3}z_2(2+2|z_1|^2-|z_2|^2),\sqrt{6}z_1(1+|z_1|^2-2|z_2|^2),-3\bar z_2z_1^3,\\&\sqrt{3}z_1^2(1+|z_1|^2-2|z_2|^2),\sqrt{3}z_1z_2(2+2|z_1|^2-|z_2|^2),3z_2^2(1+|z_1|^2))\,.\end{aligned}$$

$V(\varphi_2)=\underline{\operatorname{Im}}\,A^{\varphi_1}_{(1,0)}$ is locally spanned by ξ_2^1, ξ_2^2 and ξ_2^3, where

$$\begin{aligned}\xi_2^1=(&6\bar z_1^2,-2\sqrt{3}\bar z_1(2-|z_1|^2+2|z_2|^2),6\sqrt{3}\bar z_1^2z_2,\\&2\sqrt{3}(1+|z_2|^2)(1-2|z_1|^2+|z_2|^2),6\sqrt{3}\bar z_1^2z_2^2,-2\sqrt{6}z_2\bar z_1(2-|z_1|^2+2|z_2|^2),\\&6z_1(1+|z_2|^2)^2,2\sqrt{3}z_2(1+|z_2|^2)(1-2|z_1|^2+|z_2|^2),\\&-2\sqrt{3}z_2^2\bar z_1(2-|z_1|^2+2|z_2|^2),6\bar z_1^2z_2^3)\,,\\ \xi_2^2=(&6\bar z_1\bar z_2,-2\sqrt{3}\bar z_2(1-2|z_1|^2+|z_2|^2),-2\sqrt{3}\bar z_1(1+|z_1|^2-2|z_2|^2),\\&-2\sqrt{3}z_1\bar z_2(2-|z_1|^2+2|z_2|^2),-2\sqrt{3}z_2\bar z_1(2+2|z_1|^2-|z_2|^2),\\&\sqrt{6}(1-|z_1|^4-|z_2|^4+4|z_1|^2|z_2|^2),-6\bar z_2z_1^2(1+|z_2|^2),\\&2\sqrt{3}z_1(1+|z_1|^2-|z_2|^4+2|z_1|^2|z_2|^2),2\sqrt{3}z_2(1+|z_2|^2-|z_1|^4+2|z_1|^2|z_2|^2),\\&-6z_2^2\bar z_1(1+|z_1|^2))\,,\\ \xi_2^3=(&6\bar z_2^2,6\sqrt{3}\bar z_2^2z_1,-2\sqrt{3}\bar z_2(2+2|z_1|^2-|z_2|^2),6\sqrt{3}z_1^2\bar z_2^2,\\&2\sqrt{3}(1+|z_1|^2)(1+|z_1|^2-2|z_2|^2),-2\sqrt{6}z_1\bar z_2(2+2|z_1|^2-|z_2|^2),6z_1^3\bar z_2^2,\\&-2\sqrt{3}z_1^2\bar z_2(2+2|z_1|^2-|z_2|^2),2\sqrt{3}z_1(1+|z_1|^2)(1+|z_1|^2-2|z_2|^2),\end{aligned}$$

$6z_2(1+|z_1|^2)^2)$.

$V(\varphi_3)=\underline{\mathrm{Im}}\,A^{\varphi_2}_{(1,0)}$ is locally spanned by ξ^1_3, ξ^2_3, ξ^3_3 and ξ^4_3, where

$$\begin{aligned}
\xi^1_3=(&-6\bar z_1^3, 6\sqrt{3}\,\bar z_1^2(1+|z_2|^2), -6\sqrt{3}\,\bar z_1^3 z_2, -6\sqrt{3}\,\bar z_1(1+|z_2|^2)^2, -6\sqrt{3}\,\bar z_1^3 z_2^2,\\
&6\sqrt{6}\,z_2\bar z_1^2(1+|z_2|^2), 6(1+|z_2|^2)^3, -6\sqrt{3}\,\bar z_1 z_2(1+|z_2|^2)^2,\\
&6\sqrt{3}\,z_2^2\bar z_1^2(1+|z_2|^2), -6\bar z_1^3 z_2^3)',\\
\xi^2_3=(&-6\bar z_1^2\bar z_2, 2\sqrt{3}\,\bar z_1\bar z_2(2-|z_1|^2+2|z_2|^2), 2\sqrt{3}\,\bar z_1^2(1+|z_1|^2-2|z_2|^2),\\
&-2\sqrt{3}\,\bar z_2(1+|z_2|^2)(1-2|z_1|^2+|z_2|^2), 2\sqrt{3}\,\bar z_1^2 z_2(2+2|z_1|^2-|z_2|^2),\\
&-2\sqrt{6}\,\bar z_1(1+|z_1|^2-|z_2|^4+2|z_1|^2|z_2|^2),\\
&-6z_1\bar z_2(1+|z_2|^2)^2, 2\sqrt{3}(1+|z_2|^2)(1+|z_1|^2+|z_2|^2+3|z_1|^2|z_2|^2),\\
&-2\sqrt{3}\,z_2\bar z_1(2+2|z_1|^2+2|z_2|^2+3|z_1|^2|z_2|^2), 6\bar z_1^2 z_2^2(1+|z_1|^2)),\\
\xi^3_3=(&-6\bar z_1\bar z_2^2, 2\sqrt{3}\,\bar z_2^2(1-2|z_1|^2+|z_2|^2), 2\sqrt{3}\,\bar z_1\bar z_2(2+2|z_1|^2-|z_2|^2),\\
&2\sqrt{3}\,z_1\bar z_2^2(2-|z_1|^2+2|z_2|^2), -2\sqrt{3}\,\bar z_1(1+|z_1|^2)(1+|z_1|^2-2|z_2|^2),\\
&-2\sqrt{6}\,\bar z_2(1+|z_2|^2-|z_1|^4+2|z_1|^2|z_2|^2),\\
&6z_1^2\bar z_2^2(1+|z_2|^2), -2\sqrt{3}\,z_1\bar z_2(2+2|z_1|^2+2|z_2|^2+3|z_1|^2|z_2|^2),\\
&2\sqrt{3}(1+|z_1|^2)(1+|z_1|^2+|z_2|^2+3|z_1|^2|z_2|^2), -6\bar z_1 z_2(1+|z_1|^2)^2),\\
\xi^4_3=(&-6\bar z_2^3, -6\sqrt{3}\,\bar z_2^3 z_1, 6\sqrt{3}\,\bar z_2^2(1+|z_1|^2), -6\sqrt{3}\,z_1^2\bar z_2^3, -6\sqrt{3}\,\bar z_2(1+|z_1|^2)^2,\\
&6\sqrt{6}\,z_1\bar z_2^2(1+|z_1|^2), -6z_1^3\bar z_2^3, 6\sqrt{3}\,z_1^2\bar z_2^2(1+|z_1|^2),\\
&-6\sqrt{3}\,z_1\bar z_2(1+|z_1|^2)^2, 6(1+|z_1|^2)^3).
\end{aligned}$$

Then, we see that $A^{\varphi_3}_{(1,0)}\equiv 0$, that is, φ_3 is an anti-holomorphic map into $G_4(\boldsymbol{C}^{10})$. Since $V(\varphi_3)$ is an anti-holomorphic subbundle of $(V(\varphi_0)\oplus G'(\varphi_0))^{\perp}$, the map $\psi: \boldsymbol{CP}^2\to G_5(\boldsymbol{C}^{10})$ defined by $V(\psi)=V(\varphi_0)\oplus V(\varphi_3)$ is pluriharmonic and has ∂'-isotropy order 2. To give a pluriharmonic map into $G_2(\boldsymbol{C}^{10})$ with ∂'-isotropy order 2, we now look for an anti-holomorphic line subbundle of $V(\varphi_3)$.

Let β^0 be spanned by

$$\eta^0=(0,0,0,0,0,0,\bar z_2^3, -\sqrt{3}\,\bar z_1\bar z_2^2, \sqrt{3}\,\bar z_1^2\bar z_2, -\bar z_1^3).$$

In fact, $\eta^0=(1/36)\partial_2\partial_2\partial_2\xi^1_3=-(1/12)\partial_2\partial_2\partial_1\xi^2_3=(1/12)\partial_2\partial_1\partial_1\xi^3_3=-(1/36)\partial_1\partial_1\partial_1\xi^4_3$. Since $\partial C^\infty(V(\varphi_3))\subset C^\infty(V(\varphi_3))$ and $\partial\eta^0=0$, we see that β^0 is an anti-holomorphic line subbundle. Define φ^1 by $V(\varphi^1)=V(\varphi_0)\oplus\beta^0$. Then, φ^1 is a pluriharmonic map of $\boldsymbol{CP}^2\setminus S_{\varphi^1}$ into $G_2(\boldsymbol{C}^{10})$ with ∂'-isotropy order 2.

Next, we construct a pluriharmonic map φ^2 which has ∂'-isotropy order 1 by the backward replacement of φ^1. Since β^0 is a holomorphic and anti-holomorphic subbundle of $V(\varphi^1)$, we may take $\alpha^1=\beta^0$. A simple computation shows that $G''_{\varphi^1}(\alpha^1)$ is locally spanned by

$$\eta^1=(0,0,0,0,0,0,3|z_1|^2\bar{z}_2^2,\ -\sqrt{3}(2|z_1|^2-|z_2|^2)\bar{z}_1\bar{z}_2,\ \sqrt{3}(|z_1|^2-2|z_2|^2)\bar{z}_1^2,\ 3\bar{z}_1^3z_2)\,.$$

Define φ^2 by $V(\varphi^2)=V(\varphi_0)\oplus G''_{\varphi^1}(\alpha^1)$. Then, φ^2 is a pluriharmonic map of $CP^2\setminus S_{\varphi^2}$ into $G_2(C^{10})$ with ∂'-isotropy order 1, where $S_{\varphi^2}=S_{\varphi^1}$.

6. Pluriharmonic maps into other complex Grassmann manifolds. Let M be compact complex manifold with $c_1(M)>0$. The method and idea used to classify all pluriharmonic maps from M into $G_2(C^n)$ are also partially applicable to the case where the target is $G_3(C^n)$ or $G_4(C^n)$. There are many different cases to analyze, but the essence of the analysis is just the same as in Section 3. Therefore, for the method to increase the isotropy order of a given pluriharmonic map, we present only the algorithm.

LEMMA 6.1. *Let $\varphi: M\setminus S_\varphi\to G_3(C^n)$ be a pluriharmonic map. Assume that φ has ∂'-isotropy order r. Then, $A^3_{r,\varphi}\equiv 0$.*

(I) *If $A^2_{r,\varphi}\not\equiv 0$, set $\alpha^0=\underline{\mathrm{Im}}\,A^2_{r,\varphi}$. Then, $\alpha^0\subset\mathrm{Ker}(A^{\varphi^\perp}_{(1,0)}\circ A^{\varphi}_{(1,0)})$, and define φ^1 from φ by the forward replacement of α^0. Then, φ^1 has ∂'-isotropy order r and satisfies $A^2_{r,\varphi^1}\equiv 0$. Set $\beta^0=\underline{\mathrm{Im}}\,A_{r,\varphi^1}$ and $\delta^0=\underline{\mathrm{Im}}\,A^{G^{(r+1)}_{\varphi^1}(\beta^0),V(\varphi^1)\ominus\beta^0}_{(1,0)}$. Then, $\beta^0\oplus\delta^0\subset\mathrm{Ker}(A^{(\varphi^1)^\perp}_{(1,0)}\circ A^{\varphi^1}_{(1,0)})$. Define φ_1 from φ^1 by the forward replacement of $\beta^0\oplus\delta^0$. Then, φ_1 has ∂'-isotropy order $\geq r+1$ and satisfies $A^3_{r+1,\varphi_1}\equiv 0$.*

(II) *If $A^2_{r,\varphi}\equiv 0$, set $\alpha^0=\underline{\mathrm{Im}}\,A_{r,\varphi}$ and $\delta^0=\underline{\mathrm{Im}}\,A^{G^{(r+1)}_{\varphi}(\alpha^0),V(\varphi)\ominus\alpha^0}_{(1,0)}$. Then, $\alpha^0,\delta^0\subset\mathrm{Ker}(A^{\varphi^\perp}_{(1,0)}\circ A^{\varphi}_{(1,0)})$, $\mathrm{rank}\,\alpha^0=1, 2$, and $\mathrm{rank}\,\delta^0=0, 1$.*

(II-1) *If $\mathrm{rank}\,\alpha^0=2$, then $\delta^0=0$ and define φ_1 from φ by the forward replacement of α^0.*

(II-2) *If $\mathrm{rank}\,\alpha^0=1$ and $\delta^0=0$, then define φ_1 from φ by the forward replacement of α^0.*

(II-3) *If $\mathrm{rank}\,\alpha^0=\mathrm{rank}\,\delta^0=1$, then define φ_1 from φ by the forward replacement of $\alpha^0\oplus\delta^0$.*

Then, in any case, φ_1 has ∂'-isotropy order $\geq r+1$ and satisfies $A^3_{r+1,\varphi_1}\equiv 0$.

Moreover, for each φ_1 in (I), (II), *the following are true*:

(1) *If $A_{r+1,\varphi_1}\equiv 0$, then φ_1 is a pluriharmonic map into CP^{n-1} or $G_2(C^n)$ (the latter case occurs only for* (II-2)).

(2) *If $A^2_{r+1,\varphi_1}\equiv 0$ and $A_{r+1,\varphi_1}\not\equiv 0$, set $\mu^1=\underline{\mathrm{Im}}\,A_{r+1,\varphi_1}$ and $\delta^1=\underline{\mathrm{Im}}\,A^{G^{(r+2)}_{\varphi_1}(\mu^1),V(\varphi_1)\ominus\mu^1}_{(1,0)}$. Then, $\mu^1,\delta^1\subset\mathrm{Ker}(A^{\varphi_1^\perp}_{(1,0)}\circ A^{\varphi_1}_{(1,0)})$ and $\mathrm{rank}\,\delta^1=0, 1$ (the latter case occurs only for* (II-2)). *Define φ_2 from φ_1 by the forward replacement of $\mu^1\oplus\delta^1$. Then, φ_2 has ∂'-isotropy order $\geq r+2$ and satisfies $\mathrm{rank}\,\underline{\mathrm{Im}}\,A_{r+2,\varphi_2}=\mathrm{rank}\,V(\varphi_2)-m$, where $m=1, 2$ (the latter case occurs only for* (II-2)).

(3) *If $A^2_{r+1,\varphi_1}\not\equiv 0$, set $\alpha^1=\underline{\mathrm{Im}}\,A^2_{r+1,\varphi_1}$. Then, $\alpha^1\subset\mathrm{Ker}(A^{\varphi_1^\perp}_{(1,0)}\circ A^{\varphi_1}_{(1,0)})$, and define φ_1^1 from φ_1 by the forward replacement of α^1. Then, φ_1^1 has ∂'-isotropy order $r+1$ and satisfies $A^2_{r+1,\varphi_1^1}\equiv 0$. Set $\beta^1=\underline{\mathrm{Im}}\,A_{r+1,\varphi_1^1}$ and $\delta^1=\underline{\mathrm{Im}}\,A^{G^{(r+2)}_{\varphi_1^1}(\beta^1),V(\varphi_1^1)\ominus\beta^1}_{(1,0)}$. Then, $\beta^1,\delta^1\subset\mathrm{Ker}(A^{(\varphi_1^1)^\perp}_{(1,0)}\circ$*

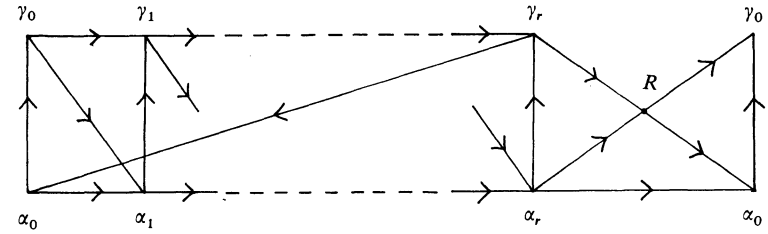


FIGURE 8.

$A^{\varphi_1^1}_{(1,0)}$). *Define* φ_2 *from* φ_1^1 *by the forward replacement of* $\beta^1 \oplus \delta^1$. *Then,* φ_2 *has* ∂'*-isotropy order* $\geq r+2$ *and satisfies* rank $\underline{\mathrm{Im}}\, A_{r+2,\varphi_2} = \mathrm{rank}\, V(\varphi_2) - 1$.

Using Lemma 6.1, we may prove the following:

PROPOSITION 6.1. *Let* $\varphi : M \setminus S_\varphi \to G_3(\boldsymbol{C}^n)$ *be a pluriharmonic map. Assume that* φ *has finite* ∂'*-isotropy order and* $n \leq 15$. *Then, there is a sequence* $\{\varphi_i\}_{i=0}^N$ *of pluriharmonic maps such that*

(1) $\varphi_0 = \varphi$, (2) $\varphi_N : M \setminus S_{\varphi_N} \to \boldsymbol{C}P^{n-1}$ *or* $G_2(\boldsymbol{C}^n)$, (3) *for* $i = 0, 1, \ldots, N-1$, *each* φ_i *has finite* ∂'*-isotropy order, and* φ_{i+1} *is obtained from* φ_i *by the forward replacement of* α^i, *where* α^i *is a holomorphic subbundle of* $V(\varphi_i)$ *contained in* $\mathrm{Ker}(A^{\varphi_i^\perp}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$.

PROOF. Construct φ_2 from φ, using Lemma 6.1. Let r be the ∂'-isotropy order of φ_2. Then, $r \geq 3$. Set $\alpha_0 = \underline{\mathrm{Im}}\, A_{r,\varphi_2}$, $\alpha_i = G^{(i)}_{\varphi_2}(\alpha_0)$ for $i = 1, \ldots, r$ and $\gamma_0 = V(\varphi_2) \ominus \alpha_0$, $\gamma_i = G^{(i)}(\varphi_2) \ominus \alpha_i$ for $i = 1, \ldots, r$. By Lemma 6.1, we have rank $\gamma_0 = m$ and rank $\alpha_0 =$ rank $V(\varphi_2) - m$, where $m = 1,2$. If $\alpha_0 = 0$, then φ_2 is a pluriharmonic map into $\boldsymbol{C}P^{n-1}$ or $G_2(\boldsymbol{C}^n)$, hence we may assume that $\alpha_0 \neq 0$. Set $R = V(\varphi_2^\perp) \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi_2))$. We have a diagram as in Figure 8.

We have two possibilities: (1) $\alpha_i = 0$ for some $1 \leq i \leq r$, (2) any α_i $(1 \leq i \leq r)$ is non-zero.

(1) Set $V(\tilde{\varphi}) = (V(\varphi_2) \ominus \alpha_0) \oplus \alpha_1$. Then, by Figure 8 we see that either $\tilde{\varphi}$ is a pluriharmonic map into $\boldsymbol{C}P^{n-1}$ or $G_2(\boldsymbol{C}^n)$, or $\tilde{\varphi}$ has ∂'-isotropy order $r+1$ and rank $\underline{\mathrm{Im}}\, A^{G^{(r+1)}(\tilde{\varphi}), V(\tilde{\varphi})}_{(1,0)} = \mathrm{rank}\, V(\tilde{\varphi}) - m$, where $m = 1, 2$.

(2) Since $n \leq 15$, one of $V(\varphi_2)$, $G^{(i)}(\varphi_2)$ $(1 \leq i \leq r)$ has rank ≤ 3 and ∂'-isotropy order r. Hence, by Lemma 6.1, either we have a pluriharmonic map into $\boldsymbol{C}P^{n-1}$ or $G_2(\boldsymbol{C}^n)$, or we have a pluriharmonic map $\tilde{\varphi}$ which has ∂'-isotropy order $r+2$ and satisfies rank $\underline{\mathrm{Im}}\, A_{r+2,\tilde{\varphi}} = \mathrm{rank}\, V(\tilde{\varphi}) - m$, where $m = 1, 2$.

Since the ∂'-isotropy order cannot be so large, repeating this procedure, we see that φ is reduced to a pluriharmonic map into $\boldsymbol{C}P^{n-1}$ or $G_2(\boldsymbol{C}^n)$, and each φ_i in the sequence has the desired properties by Proposition 2.3. q.e.d.

Next, we present an algorithm to increase the isotropy order of a given pluriharmonic map into $G_4(\boldsymbol{C}^n)$. To state the algorithm, we need the following:

PROPOSITION 6.2 ([cf. O-U2]). *Let* $\varphi : M \setminus S_\varphi \to G_k(\boldsymbol{C}^n)$ *be a pluriharmonic map* *with ∂'-isotropy order r. Assume that $A^2_{r,\varphi}\equiv 0$ and* $\operatorname{rank}\underline{\operatorname{Im}} A_{r,\varphi}=1$. *Then, there is a holomorphic subbundle τ of $V(\varphi)$ with $\tau\subset\operatorname{Ker}(A^{\varphi^\perp}_{(1,0)}\circ A^{\varphi}_{(1,0)})$ such that φ_1 defined from φ by the forward replacement of τ has ∂'-isotropy order $\geq r+1$.*

The method is to construct a pluriharmonic map $\tilde\varphi$ satisfying the conditions in Proposition 6.2 from a given pluriharmonic map φ.

LEMMA 6.2. *Let $\varphi\colon M\setminus S_\varphi\to G_4(C^n)$ be a pluriharmonic map. Assume that φ has ∂'-isotropy order r. Then $A^4_{r,\varphi}\equiv 0$. There are three possibilities*: (I) $A^3_{r,\varphi}\not\equiv 0$, (II) $A^3_{r,\varphi}\equiv 0$ *and* $A^2_{r,\varphi}\not\equiv 0$, (III) $A^2_{r,\varphi}\equiv 0$.

In each case, we use the following notation for simplicity:

$$A=A^{\varepsilon^r,\mu^0}_{(1,0)}\circ A^{\varepsilon^{r-1},\varepsilon^r}_{(1,0)}\circ\cdots\circ A^{\varepsilon^0,\varepsilon^1}_{(1,0)}\,,\qquad B=A^{\mu^{r+1},\varepsilon^0}_{(1,0)}\circ A^{\mu^r,\mu^{r+1}}_{(1,0)}\circ\cdots\circ A^{\mu^0,\mu^1}_{(1,0)}\,.$$

(I) *Set $\alpha^0=\underline{\operatorname{Im}} A^3_{r,\varphi}$. Then, $\alpha^0\subset\operatorname{Ker}(A^{\varphi^\perp}_{(1,0)}\circ A^{\varphi}_{(1,0)})$, and define φ^1 from φ by the forward replacement of α^0. Then, φ^1 has ∂'-isotropy order r and satisfies $A^3_{r,\varphi^1}\equiv 0$. Set $\beta^0=\underline{\operatorname{Im}} A^2_{r,\varphi^1}$. Then, $\beta^0\subset\operatorname{Ker}(A^{(\varphi^1)^\perp}_{(1,0)}\circ A^{\varphi^1}_{(1,0)})$, and define φ^2 from φ^1 by the forward replacement of β^0. Then, φ^2 has ∂'-isotropy order r and satisfies $A^2_{r,\varphi^2}\equiv 0$. Set $\mu^0=\underline{\operatorname{Im}} A_{r,\varphi^2}$. Then, $\mu^0\subset\operatorname{Ker}(A^{(\varphi^2)^\perp}_{(1,0)}\circ A^{\varphi^2}_{(1,0)})$, and set $\mu^i=G^{(i)}_{\varphi^2}(\mu^0)$ for $i=1,\dots,r+1$, $\varepsilon^0=V(\varphi^2)\ominus\mu^0$, $\varepsilon^i=G^{(i)}(\varphi^2)\ominus\mu^i$ for $i=1,\dots,r$, $\eta=\underline{\operatorname{Im}} B$ and $\nu=\underline{\operatorname{Im}}(A\circ B)$. Then,* $\operatorname{rank}\eta=\operatorname{rank}\varepsilon^0-1$, $\operatorname{rank}\nu=\operatorname{rank}\mu^0-1$ *and $\nu\subset\operatorname{Ker}B$. Define $\tilde\varphi$ from φ^2 by the forward replacement of ν. Then, $\tilde\varphi$ satisfies the conditions in Proposition* 6.2.

(II) *Set $\alpha^0=\underline{\operatorname{Im}} A^2_{r,\varphi}$, $\beta^0=\underline{\operatorname{Im}} A_{r,\varphi}\ominus\alpha^0$ and $\gamma^0=V(\varphi)\ominus(\alpha^0\oplus\beta^0)$. Then, there are three possibilities*: (II-1) $\operatorname{rank}\alpha^0=\operatorname{rank}\beta^0=1$, (II-2) $\operatorname{rank}\alpha^0=\operatorname{rank}\gamma^0=1$, (II-3) $\operatorname{rank}\beta^0=\operatorname{rank}\gamma^0=1$.

(II-1) (1) *If $A^{G^{(r+1)}_{\varphi}(\alpha^0),\gamma^0}_{(1,0)}\equiv 0$, define $\tilde\varphi$ from φ by the forward replacement of α^0. Then $\tilde\varphi$ satisfies the conditions in Proposition* 6.2.

(2) *Otherwise, set $\delta^0=\underline{\operatorname{Im}} A^{G^{(r+1)}_{\varphi}(\alpha^0),\gamma^0}_{(1,0)}$ and define φ^1 from φ by the forward replacement of α^0. Then φ^1 has ∂'-isotropy order r and satisfies $A^2_{r,\varphi^1}\equiv 0$. Set $\mu^0=\underline{\operatorname{Im}} A_{r,\varphi^1}$, $\mu^i=G^{(i)}_{\varphi^1}(\mu^0)$ for $i=1,\dots,r+1$, $\varepsilon^0=V(\varphi^1)\ominus\mu^0$, $\varepsilon^i=G^{(i)}(\varphi^1)\ominus\mu^i$ for $i=1,\dots,r$, $\eta=\underline{\operatorname{Im}} B$ and $\nu=\underline{\operatorname{Im}}(A\circ B)$. Then* $\operatorname{rank}\eta=\operatorname{rank}\varepsilon^0-1$, $\operatorname{rank}\nu=1$ *and $\nu\subset\operatorname{Ker}B$. Define $\tilde\varphi$ from φ^1 by the forward replacement of ν.* Then *$\tilde\varphi$ satisfies the conditions in Proposition* 6.2.

(II-2) *Define φ^1 from φ by the forward replacement of α^0. Then, φ^1 has ∂'-isotropy order r and satisfies $A^2_{r,\varphi^1}\equiv 0$. Set $\mu^0=\underline{\operatorname{Im}} A_{r,\varphi^1}$, $\mu^i=G^{(i)}_{\varphi^1}(\mu^0)$ for $i=1,\dots,r+1$, $\varepsilon^0=V(\varphi^1)\ominus\mu^0$, $\varepsilon^i=G^{(i)}(\varphi^1)\ominus\mu^i$ for $i=1,\dots,r$ and $\eta=\underline{\operatorname{Im}} B$. Then,* $\operatorname{rank}\eta=\operatorname{rank}\varepsilon^0-1$ *and $(A\circ B)^2\equiv 0$.*

(1) *If $\eta\subset\operatorname{Ken}A$, set $\nu=\mu^0\oplus\eta$. Then, $\nu\subset\operatorname{Ker}(A^{(\varphi^1)^\perp}_{(1,0)}\circ A^{\varphi^1}_{(1,0)})$,* $\operatorname{rank}\nu=\operatorname{rank}V(\varphi^1)-1$, *and φ_1 defined from φ^1 by the forward replacement of ν has ∂'-isotropy order $\geq r+1$ and satisfies* $\operatorname{rank}\underline{\operatorname{Im}} A_{r+1,\varphi_1}=\operatorname{rank}V(\varphi_1)-1$.

(2) *If $\eta\subset\operatorname{Ker}A$, set $\nu=\underline{\operatorname{Im}}(A\circ B)$. Then,* $\operatorname{rank}\nu=1$ *and $\nu\subset\operatorname{Ker}B$. Define $\tilde\varphi$ from φ^1 by the forward replacement of ν. Then, $\tilde\varphi$ satisfies the conditions in Proposition* 6.2.

(II-3) *Define* $\tilde{\varphi}$ *from* φ *by the forward replacement of* α^0. *Then,* $\tilde{\varphi}$ *satisfies the conditions in Proposition* 6.2.

(III) *Set* $\mu^0 = \underline{\mathrm{Im}}\, A_{r,\varphi}$. *Then,* $\mu^0 \subset \mathrm{Ker}\, A_{r,\varphi}$. *Set* $\mu^i = G_\varphi^{(i)}(\mu^0)$ *for* $i = 1, \dots, r+1$, $\varepsilon^0 = V(\varphi) \ominus \mu^0$ *and* $\varepsilon^i = G^{(i)}(\varphi) \ominus \mu^i$ *for* $i = 1, \dots, r$. *Then there are three possibilities*:

(III-1) $\mathrm{rank}\, \mu^0 = 1$, (III-2) $\mathrm{rank}\, \mu^0 = 2$, (III-3) $\mathrm{rank}\, \mu^0 = 3$.

(III-1) *In this case,* φ *itself satisfies the conditions in Proposition* 6.2. *Set* $\tilde{\varphi} = \varphi$.

(III-2) *We have* $\mathrm{rank}\, \underline{\mathrm{Im}}\, A_{(1,0)}^{\mu^{r+1},\varepsilon^0} = 0, 1$.

(1) *If* $A_{(1,0)}^{\mu^{r+1},\varepsilon^0} \equiv 0$, *then define* φ_1 *from* φ *by the forward replacement of* μ^0. *Then* φ_1 *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $\mathrm{rank}\, \underline{\mathrm{Im}}\, A_{r+1,\varphi_1} = \mathrm{rank}\, V(\varphi_1) - m$, *where* $m = 1, 2$.

(2) *Otherwise, set* $\eta = \underline{\mathrm{Im}}\, A_{(1,0)}^{\mu^{r+1},\varepsilon^0}$. *Then* $\mathrm{rank}\, \eta = 1$ *and* $(A \circ B)^2 \equiv 0$. *If* $A \circ B \equiv 0$, *set* $\nu = \mu^0 \oplus \eta$. *Then* $\nu \subset \mathrm{Ker}(A_{(1,0)}^{\varphi^\perp} \circ A_{(1,0)}^{\varphi})$, $\mathrm{rank}\, \nu = \mathrm{rank}\, V(\varphi) - 1$, *and* φ_1 *defined from* φ *by the forward replacement of* ν *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $\mathrm{rank}\, \underline{\mathrm{Im}}\, A_{r+1,\varphi_1} = \mathrm{rank}\, V(\varphi_1) - 1$. *If* $A \circ B \not\equiv 0$, *set* $\nu = \underline{\mathrm{Im}}(A \circ B)$. *Then* $\mathrm{rank}\, \nu = 1$ *and* $\nu \subset \mathrm{Ker}\, B$. *Define* $\tilde{\varphi}$ *from* φ *by the forward replacement of* ν. *Then,* $\tilde{\varphi}$ *satisfies the conditions in Proposition* 6.2.

(III-3) *We have* $A_{(1,0)}^{\mu^{r+1},\varepsilon^0} \equiv 0$. Define φ_1 *from* φ *by the forward replacement of* μ^0. *Then* φ_1 *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $\mathrm{rank}\, \underline{\mathrm{Im}}\, A_{r+1,\varphi_1} = \mathrm{rank}\, V(\varphi_1) - 1$.

Moreover, for each $\tilde{\varphi}$, *there is a holomorphic subbundle* τ *of* $\tilde{\varphi}$ *with* $\tau \subset \mathrm{Ker}(A_{(1,0)}^{\tilde{\varphi}^\perp} \circ A_{(1,0)}^{\tilde{\varphi}})$ *and* $\mathrm{rank}\, \tau = \mathrm{rank}\, V(\tilde{\varphi}) - m$ *such that* φ_1 *defined from* $\tilde{\varphi}$ *by the forward replacement of* τ *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $\mathrm{rank}\, \underline{\mathrm{Im}}\, A_{r+1,\varphi_1} = \mathrm{rank}\, V(\varphi_1) - m$, *where* $m = 1, 2, 3$.

Using Lemma 6.2, we obtain:

PROPOSITION 6.3. *Let* $\varphi: M \setminus S_\varphi \to G_4(C^n)$ *be a pluriharmonic map. Assume that* φ *has finite* ∂'*-isotropy order and* $n \leq 14$. *Then, there is a sequence* $\{\varphi_i\}_{i=0}^N$ *of pluriharmonic maps such that*

(1) $\varphi_0 = \varphi$, (2) $\varphi_N: M \setminus S_{\varphi_N} \to G_m(C^n)$ *with* $m = 1, 2, 3$, (3) *for* $i = 0, 1, \dots, N-1$, *each* φ_i *has finite* ∂'*-isotropy order, and* φ_{i+1} *is obtained from* φ_i *by the forward replacement of* α^i, *where* α^i *is a holomorphic subbundle of* $V(\varphi_i)$ *contained in* $\mathrm{Ker}(A_{(1,0)}^{\varphi_i^\perp} \circ A_{(1,0)}^{\varphi_i})$.

This follows from the arguments similar to those of Proposition 6.1. For the inverse procedure, we have the following:

THEOREM 6.1. *Let* $\varphi: M \setminus S_\varphi \to G_k(C^n)$ *be a pluriharmonic map with finite* ∂'*-isotropy order, where* M *is a compact complex manifold with* $c_1(M) > 0$. *Assume that* $k = 3$ (*resp.* $k = 4$) *and* $n \leq 15$ (*resp.* $n \leq 14$). *Then, there is a sequence* $\{\varphi^i\}_{i=0}^N$ *of pluriharmonic maps such that*

(1) $\varphi^N = \varphi$, (2) $\varphi^0: M \setminus S_{\varphi^0} \to G_t(C^n)$ *with* $1 \leq t \leq k-1$, *and* φ^1 *is obtained from* φ^0 *by the backward extension so that* φ^1 *has finite* ∂'*-isotropy order,* (3) *for* $i = 1, \dots, N-1$, *each* φ^{i+1} *has finite* ∂'*-isotropy order, and* φ^{i+1} *is obtained from* φ^i *by the backward replacement and the backward extension.*

REMARK. If we do not require the result such as the one in (2) of Theorem 6.1, the restriction on n may be relaxed to the condition that $n \leq 20$ *for* $k=3$ and $n \leq 15$ for $k=4$.

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DEPARTMENT OF MATHEMATICS
SCHOOL OF MEDICINE
NIHON UNIVERSITY
ITABASHI-KU, TOKYO 173
JAPAN