J. Math. Soc. Japan
Vol. 38, No. 3, 1986

Spectral geometry of Kaehler submanifolds of a complex projective space

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(Received June 29, 1984)
(Revised Dec. 22, 1984)

§ 0. Introduction.

Let $X: M \to E^N$ be an isometric immersion of a compact Riemannian manifold into an N-dimensional Euclidean space. Then X can be decomposed as $X = \sum_{k \in N} X_k$, where X_k is the k-th eigenfunction of the Laplacian of M (for details, see § 2). We say that the immersion is of order $\{k_1, k_2, k_3\}$ (resp. $\{k_1, k_2\}$ and k_1) if $X = X_0 + X_{k_1} + X_{k_2} + X_{k_3}$ (resp. $X = X_0 + X_{k_1} + X_{k_2}$ and $X = X_0 + X_{k_1}$), where X_0 is a constant mapping and $X_{k_1}, X_{k_2}, X_{k_3} \neq 0$ and $0 < k_1 < k_2 < k_3$.

Let $F: CP^m \to E^N$ be the standard isometric imbedding of a complex projective space into an N-dimensional Euclidean space (for details, see § 1), and let $A: M \to CP^m$ be an isometric immersion of a compact Kaehler manifold into an m-dimensional complex projective space. Then A is said to be of order $\{k_1, k_2, k_3\}$ (resp. $\{k_1, k_2\}$ and k_1) if the immersion $F \circ A$ is of order $\{k_1, k_2, k_3\}$ (resp. $\{k_1, k_2\}$ and k_1). A totally geodesic Kaehler submanifold of CP^m is of order 1. Moreover there does not exist any compact Kaehler submanifold of order k_1 ($k_1 \geqq 2$) (see, [**8**], [**9**]), and a compact Kaehler submanifold is of order 1 if and only if it is totally geodesic. A. Ros ([**9**]) proved that Einstein Kaehler submanifolds with parallel second fundamental form except $E_6/Spin(10) \times T$ in a complex projective space are of order $\{1, 2\}$, and he characterized them by their spectra in the class of compact Kaehler submanifolds in a complex projective space. In § 4, we calculate the eigenvalues of the Laplacians of $E_6/Spin(10) \times T$ and $E_7/E_6 \times T$. Consequently, we see that $E_6/Spin(10) \times T$ is of order $\{1, 2\}$, and we can say that a compact Kaehler submanifold different from a totally geodesic Kaehler submanifold in a complex projective space is of order $\{1, 2\}$ if it is Einstein and has parallel second fundamental form (Proposition 3). Moreover we can characterize $E_6/Spin(10) \times T$ by its spectrum in the class of compact Kaehler submanifolds in a complex projective space (Proposition 4).

Next, by applying Ros' method, we prove that $CP^n(1/3)$ and compact irreducible Hermitian symmetric spaces of rank 3 in $CP^{n+p}(1)$ are all of order $\{1, 2, 3\}$ (Proposition 5), where $CP^m(c)$ denotes an m-dimensional complex projective space of holomorphic sectional curvature c.

The main result of this paper is the following.

THEOREM. *Let M be an n-dimensional compact Einstein Kaehler submanifold immersed in $CP^{n+p}(1)$, and let $\tilde{M}$ be one of the Hermitian symmetric submanifolds given in Tables 2 and 3 (i.e., compact Einstein Hermitian symmetric submanifolds of degree 3).*

If $\mathrm{Spec}(M)=\mathrm{Spec}(\tilde{M})$, *then M is congruent to $\tilde{M}$.*

The author wishes to thank Professors K. Ogiue and N. Ejiri for many valuable comments and encouragements.

§ 1. Preliminaries.

Let $\mathrm{HM}(m+1)=\{A\in gl(m+1, \boldsymbol{C}) \mid \bar{A}={}^tA\}$ be the space of $(m+1)\times(m+1)$-Hermitian matrices. We define on $\mathrm{HM}(m+1)$ an inner product g by

$$g(A, B)=2\,\mathrm{tr}\,AB \qquad \text{for} \quad A, B\in \mathrm{HM}(m+1).$$

We consider the submanifold $CP^m=\{A\in \mathrm{HM}(m+1) \mid AA=A, \mathrm{tr}\,A=1\}$. It is known that CP^m, with the metric induced from g on $\mathrm{HM}(m+1)$, is isometric to the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. The tangent space and the normal space at any point A of CP^m are given respectively by

$$T_A(CP^m)=\{X\in \mathrm{HM}(m+1) \mid XA+AX=X\},$$

$$T_A^{\perp}(CP^m)=\{Z\in \mathrm{HM}(m+1) \mid ZA=AZ\}.$$

Let D, $\tilde{\nabla}$, $\tilde{\sigma}$, $\tilde{\nabla}^{\perp}$, $\tilde{A}$, $\tilde{H}$ be the Riemannian connection of $\mathrm{HM}(m+1)$, the induced connection in CP^m, the second fundamental form of the immersion, the normal connection, the Weingarten endomorphism, the mean curvature vector of CP^m in $\mathrm{HM}(m+1)$, respectively.

A. Ros [8, 9] obtained the following facts.

(1.1) $$\tilde{\sigma}(X, Y)=(XY+YX)(I-2A),$$

(1.2) $$\tilde{A}_Z X=(XZ-ZX)(I-2A),$$

(1.3) $$\tilde{H}=\frac{1}{2m}[I-(m+1)A],$$

(1.4) $$JX=\sqrt{-1}(I-2A)X,$$

(1.5) $$\tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y), \qquad \tilde{\nabla}\tilde{\sigma}=0,$$

(1.6) $$g(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W))=\frac{1}{2}g(X, Y)g(V, W)+\frac{1}{4}\{g(X, W)g(Y, V)$$

$$+g(X, V)g(Y, W)+g(X, JW)g(Y, JV)$$
$$+g(X, JV)g(Y, JW)\},$$

(1.7) $$\tilde{A}_{\tilde{\sigma}(X,Y)}V=\frac{1}{2}g(X, Y)V+\frac{1}{4}\{g(Y, V)X+g(X, V)Y+g(JY, V)JX+g(JX, V)JY\},$$

(1.8) $$g(\tilde{\sigma}(X, Y), I)=0, \quad g(\tilde{\sigma}(X, Y), A)=-g(X, Y),$$

where I is the $(m+1)\times(m+1)$-identity matrix, J is the complex structure of CP^m, $X, Y, V, W\in T_A(CP^m)$ and $Z\in T_A^{\perp}(CP^m)$.

§ 2. The order of an immersion.

Let $X: M^n\rightarrow E^N$ be an isometric immersion of an n-dimensional compact Riemannian manifold into the N-dimensional Euclidean space. Let Δ be the Laplacian of M acting on differentiable functions and $\mathrm{Spec}(M)=\{0<\lambda_1=\cdots=\lambda_1<\lambda_2=\cdots=\lambda_2<\cdots\}$ be the spectrum of Δ. Then we have the orthogonal decomposition $X=\sum_k X_k$, $k\in \boldsymbol{N}$, where $X_k: M\rightarrow E^N$ is a differentiable mapping satisfying $\Delta X_k=\lambda_k X_k$, and the addition is convergent, componentwise, for the L^2-topology on $C^\infty(M)$.

We have the relations

(2.1) $$\Delta X=-nH=\sum_{k\geq 1}\lambda_k X_k,$$

(2.2) $$\Delta^2 X=-n\Delta H=\sum_{k\geq 1}\lambda_k^2 X_k,$$

(2.3) $$\Delta^3 X=-n\Delta^2 H=\sum_{k\geq 1}\lambda_k^3 X_k,$$

where H is the mean curvature vector of M in E^N.

Let $k_1, k_2, k_3\in \boldsymbol{N}$ with $0<k_1<k_2<k_3$. We say that the immersion X is of *order* k_1 (resp. $\{k_1, k_2\}$ and $\{k_1, k_2, k_3\}$) if $X=X_0+X_{k_1}$ (resp. $X_0+X_{k_1}+X_{k_2}$ and $X_0+X_{k_1}+X_{k_2}+X_{k_3}$) and $X_{k_1}, X_{k_2}, X_{k_3}\neq 0$.

§ 3. Kaehler submanifolds.

Let M^n be an n-dimensional compact Kaehler submanifold immersed in the $(n+p)$-dimensional complex projective space CP^{n+p}, and let $A: M^n\rightarrow CP^{n+p}$ be the immersion. Let $E_1, \cdots, E_n, E_{1*}=JE_1, \cdots, E_{n*}=JE_n, \xi_1, \cdots, \xi_p, \xi_{1*}=J\xi_1, \cdots, \xi_{p*}=J\xi_p$ be a local field of orthonormal frames of CP^{n+p}, such that, restricted to M, $E_1, \cdots, E_n, E_{1*}, \cdots, E_{n*}$ are tangent to M. Let ∇, σ, $\nabla^{\perp}$ and A be the Riemannian connection, the second fundamental form, the normal connection and the Weingarten endomorphism of M in CP^{n+p} respectively, and H the mean curvature vector of M in $\mathrm{HM}(n+p+1)$.

Throughout this paper, we use the following convention on the range of indices: $i, j, k, l, \cdots=1, \cdots, n, 1^*, \cdots, n^*$, $\lambda, \mu, \cdots=1, \cdots, p, 1^*, \cdots, p^*$, $A, B, C, \cdots=1, \cdots, n, n+1, \cdots, n+p$, $a, b, c, \cdots=1, \cdots, n$, $\alpha, \beta, \gamma, \cdots=1, \cdots, p$. Then, the immersion X is of order k_1 if and only if M^n is totally geodesic and the immersion X is of order $\{k_1, k_2\}$ if and only if

$$\Delta H=(\lambda_{k_1}+\lambda_{k_2})H+\frac{\lambda_{k_1}\lambda_{k_2}}{2n}(X-X_0) \tag{3.1}$$

(see [9]), and in the same way as in p. 440 of [9] we can see that the immersion X is of order $\{k_1, k_2, k_3\}$ if and only if

$$\Delta^2 H=(\lambda_{k_1}+\lambda_{k_2}+\lambda_{k_3})\Delta H-(\lambda_{k_1}\lambda_{k_2}+\lambda_{k_2}\lambda_{k_3}+\lambda_{k_3}\lambda_{k_1})H-\frac{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}{2n}(X-X_0). \tag{3.2}$$

We prepare the following Lemma.

LEMMA 1 (A. Ros [9]).

$$H=\frac{1}{2n}\sum_i \tilde{\sigma}(E_i, E_i), \tag{3.3}$$

$$\Delta H=(n+1)H+\frac{1}{n}\sum_{i,j}\tilde{\sigma}(A_{\sigma(E_i,E_j)}E_i, E_j) -\frac{1}{n}\sum_{i,j}\tilde{\sigma}(\sigma(E_i, E_j), \sigma(E_i, E_j)). \tag{3.4}$$

This is obtained by using (1.7) and the fact that M is minimal in CP^{n+p} and that CP^{n+p} has parallel second fundamental form.

The normal space of M in CP^{n+p} at x is denoted by $T_x^{\perp}(M)$. We define the tensor $T: T_x^{\perp}\times T_x^{\perp}\to \boldsymbol{R}$ by

$$T(\xi, \eta)=\operatorname{tr} A_\xi A_\eta \qquad \text{for all} \quad \xi, \eta\in T_x^{\perp}(M). \tag{3.5}$$

Then, A. Ros [9] obtained the following result.

PROPOSITION 1. *Let M be an n-dimensional compact Kaehler submanifold in CP^{n+p} such that the immersion $A: M\to CP^{n+p}$ is full. Then M is a submanifold of order $\{k_1, k_2\}$ in* $\mathrm{HM}(n+p+1)$ *if and only if M is an Einstein submanifold with $T=kg|_{T^{\perp}\times T^{\perp}}$ for some real number k.*

If the immersion is full, the constant part X_0 of X is given by $X_0=(1/(n+p+1))I$ (see [9]), where I is the $(n+p+1)\times(n+p+1)$-identity matrix.

§4. Computation of eigenvalues of Δ.

Let (G, K) be a Riemannian symmetric pair. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then we have the canonical decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$. Let $\mathfrak{a}$ be a Cartan subalgebra of (G, K), i.e., a maximal Abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$, and let $\mathfrak{t}$ be a maximal Abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we have the direct sum decomposition $\mathfrak{t}=\mathfrak{a}+\mathfrak{b}$. We define the involution S by

$$S(H_1+H_2)=-H_1+H_2, \qquad H_1\in\mathfrak{b},\ H_2\in\mathfrak{a},$$

and define $\bar{H}$ by

$$\bar{H}=\frac{1}{2}(H+S(H)), \qquad H\in\mathfrak{t}.$$

Let $\Sigma(G)$ be the set of all roots of G with respect to $\mathfrak{t}$, and define $\Sigma_0(G)$, $\Sigma(G, K)$, $\Sigma^+(G, K)$ by

$$\Sigma_0(G)=\Sigma(G)\cap\mathfrak{b}, \qquad \Sigma(G, K)=\{\bar{\alpha}\ ;\ \alpha\in\Sigma(G)-\Sigma_0(G)\},$$
$$\Sigma^+(G, K)=\{\gamma\in\Sigma(G, K)\ ;\ \gamma>0\},$$

respectively. Next, we define $\Gamma(G)$, $Z(G)$, $D(G)$, $\Gamma(G, K)$, $Z(G, K)$, $D(G, K)$ by

$$\Gamma(G)=\{H\in\mathfrak{t}\ ;\ \exp H=e\in T\},$$
$$Z(G)=\{\lambda\in\mathfrak{t}\ ;\ (\lambda, H)\in\boldsymbol{Z} \text{ for all } H\in\Gamma(G)\},$$
$$D(G)=\{\lambda\in Z(G)\ ;\ (\lambda, H)\geqq 0\},$$
$$\Gamma(G, K)=\{H\in\mathfrak{a}\ ;\ \exp H\in K\},$$
$$Z(G, K)=\{\lambda\in\mathfrak{a}\ ;\ (\lambda, H)\in\boldsymbol{Z} \text{ for all } H\in\Gamma(G, K)\},$$
$$D(G, K)=\{\lambda\in Z(G, K)\ ;\ (\lambda, \gamma)\geqq 0 \text{ for all } \gamma\in\Sigma^+(G, K)\},$$

respectively, where T is the maximal torus generated by $\mathfrak{t}$, and e is the identity, and $(\ ,\ )$ is the inner product on $\mathfrak{t}$.

Let $\Pi(G)=\{\alpha_1, \cdots, \alpha_l\}$ be the fundamental root system, and let $N_1, \cdots, N_l$ be the fundamental weights of $\mathfrak{g}$ defined by

$$\frac{2(N_i, \alpha_j)}{(\alpha_j, \alpha_j)}=\delta_{ij} \qquad \text{for} \quad N_i\in\mathfrak{t},$$

where $l=\text{rank}(G)$. Let M_i be the fundamental weights of $(\mathfrak{g}, \mathfrak{k})$ defined by

$$M_i=\begin{cases} 2N_i, & \text{if} \quad p\alpha_i=\alpha_i,\ (\alpha_i, \Pi_0(G))=\{0\} \\ N_i, & \text{if} \quad p\alpha_i=\alpha_i,\ (\alpha_i, \Pi_0(G))\neq\{0\} \\ N_i+N_j, & \text{if} \quad p\alpha_i=\alpha_j,\ \alpha_i\neq\alpha_j, \end{cases}$$

where $\Pi_0(G)=\Pi(G)\cap\Sigma_0(G)$ and p is the Satake involution. We put $\delta(G)=\sum_i N_i$.

We review the following facts (see [**12**]).

FACT 1. *Let (G, K) be a compact symmetric pair such that G/K is simply-*

connected. Then

$$D(G, K)=\left\{\sum_{i=1}^{l} m_i M_i\ ;\ m_i \in \boldsymbol{Z},\ m_i \geqq 0\ (1 \leqq i \leqq l)\right\}.$$

FACT 2. *Let ρ be a spherical representation of G with respect to K. Then the highest weight $\lambda(\rho)$ of ρ with respect to $\mathfrak{t}$ belongs to $D(G, K)$.*

FACT 3. *The mapping $\rho \rightarrow \lambda(\rho)$ is bijective.*

Now we can compute the eigenvalues of Δ for $E_6/Spin(10)\times T$ and $E_7/E_6\times T$.

i) $E_6/Spin(10)\times T$: We put $G=E_6$ and $K=Spin(10)\times T$. The fundamental roots are given by (see [2])

$$\alpha_1=\frac{1}{2}(e_1+e_8)-\frac{1}{2}(e_2+\cdots+e_7), \qquad \alpha_2=e_1+e_2,$$

$$\alpha_3=e_2-e_1, \qquad \alpha_4=e_3-e_2, \qquad \alpha_5=e_4-e_3, \qquad \alpha_6=e_5-e_4,$$

where $e_i=(0, \cdots, 0, \overset{i}{\breve{1}}, 0, \cdots, 0)\in \boldsymbol{R}^8$ for $i=1, \cdots, 8$. The fundamental weights of $\mathfrak{g}$ are given by

$$N_1=\frac{2}{3}(e_8-e_7-e_6),$$

$$N_2=\frac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7+e_8),$$

$$N_3=\frac{5}{6}(e_8-e_7-e_6)+\frac{1}{2}(-e_1+e_2+e_3+e_4+e_5),$$

$$N_4=e_3+e_4+e_5-e_6-e_7+e_8,$$

$$N_5=\frac{2}{3}(e_8-e_7-e_6)+e_4+e_5,$$

$$N_6=\frac{1}{3}(e_8-e_7-e_6)+e_5,$$

and

$$\delta(G)=\sum_i N_i=e_2+2e_3+3e_4+4e_5+4(e_8-e_7-e_6).$$

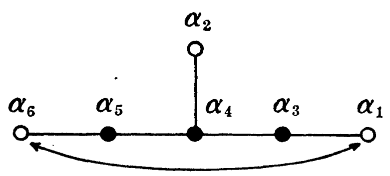

Diagram 1.

From diagram 1, the fundamental weights of $(\mathfrak{g}, \mathfrak{k})$ are given by

$$M_1 = N_1+N_6 = e_8-e_7-e_6+e_5,$$

$$M_2 = N_2 = \frac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7+e_8).$$

It follows from Facts 1, 2 and 3 that $\lambda(\rho)=m_1M_1+m_2M_2$. Therefore the Freudenthal's formula implies that the eigenvalue A_ρ of the Casimir operator of an irreducible representation ρ is given by

$$A_\rho = \frac{1}{2}(\lambda(\rho)+2\delta(G), \lambda(\rho))$$
$$= 2m_1(m_1+m_2+8)+m_2(m_2+11).$$

Since the eigenvalues $0<\lambda_1<\lambda_2<\cdots$ of Δ are given by A_ρ's, we see that

$$\lambda_1 = 12 \qquad (m_1=0,\ m_2=1),$$
$$\lambda_2 = 18 \qquad (m_1=1,\ m_2=0),$$
$$\cdots\cdots\cdots\cdots$$

ii) $E_7/E_6\times T$: We put $G=E_7$ and $K=E_6\times T$. The fundamental roots are given by

$$\alpha_1 = \frac{1}{2}(e_1+e_8)-\frac{1}{2}(e_2+e_3+e_4+e_5+e_6+e_7),$$

$$\alpha_2 = e_1+e_2, \qquad \alpha_3 = e_2-e_1, \qquad \alpha_4 = e_3-e_2,$$

$$\alpha_5 = e_4-e_3, \qquad \alpha_6 = e_5-e_4, \qquad \alpha_7 = e_6-e_5.$$

The fundamental weights of $\mathfrak{g}$ are given by

$$N_1 = e_8-e_7,$$

$$N_2 = \frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6-2e_7+2e_8),$$

$$N_3 = \frac{1}{2}(-e_1+e_2+e_3+e_4+e_5+e_6-3e_7+3e_8),$$

$$N_4 = e_3+e_4+e_5+e_6+2(e_8-e_7),$$

$$N_5 = \frac{1}{2}(2e_4+2e_5+2e_6+3e_8-3e_7),$$

$$N_6 = e_5+e_6-e_7+e_8,$$

$$N_7 = e_6+\frac{1}{2}(e_8-e_7),$$

and

$$2\delta(G)=2e_2+4e_3+6e_4+8e_5+10e_6-17e_7+17e_8.$$

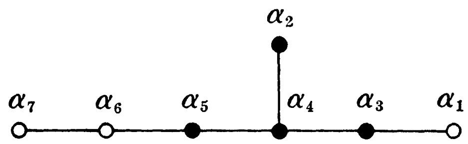

Diagram 2.

From diagram 2, the fundamental weights of $(\mathfrak{g}, \mathfrak{k})$ are given by

$$M_1 = N_1 = e_8 - e_7,$$
$$M_6 = N_6 = e_5 + e_6 - e_7 + e_8,$$
$$M_7 = 2N_7 = 2e_6 + e_8 - e_7.$$

Hence the highest weight is given by $\lambda(\rho) = m_1 M_1 + m_2 M_6 + m_3 M_7$, where $m_1, m_2, m_3 \in \boldsymbol{Z}$, $m_1, m_2, m_3 \geqq 0$. Therefore the Freudenthal's formula implies that

$$A_\rho = \frac{1}{2}(\lambda(\rho) + 2\delta(G), \lambda(\rho))$$
$$= m_1^2 + 2m_2^2 + 3m_3^2 + 2m_1m_2 + 4m_2m_3 + 2m_3m_1 + 17m_1 + 26m_2 + 27m_3.$$

Thus we see that the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ of Δ are given by

$$\lambda_1 = 18 \quad (m_1 = 1,\ m_2 = m_3 = 0),$$
$$\lambda_2 = 28 \quad (m_1 = m_3 = 0,\ m_2 = 1),$$
$$\lambda_3 = 30 \quad (m_1 = m_2 = 0,\ m_3 = 1),$$

....

§ 5. Spectral geometry for Kaehler submanifolds I.

First we state the following.

LEMMA 2 ([**9**]).

(5.1) $$g(A, A) = 2,$$

(5.2) $$g(A, H) = -1,$$

(5.3) $$g(A, \Delta H) = -(n+1),$$

(5.4) $$g(H, H) = \frac{n+1}{2n},$$

(5.5) $$g(H, \Delta H) = \frac{(n+1)^2}{2n} + \frac{1}{2n^2}\|\sigma\|^2,$$

(5.6) $$g(\Delta H, \Delta H) = \frac{(n+1)^3}{2n} + \frac{n+1}{n^2}\|\sigma\|^2 + \frac{1}{n^2}\|T\|^2 + \frac{1}{n^2}\mathrm{tr}(\sum_\lambda A_\lambda^2)^2,$$

where $A_\lambda = A_{\xi_\lambda}$.

Note that $\int_M g(X_r, X_s)=0$ for $r\neq s$, and put $a_k=\int_M g(X_k, X_k)$. Then from (2.1) and (2.2) we have

$$-2n\int_M g(X, H) = \sum_{k\geqq 1}\lambda_k a_k,$$

$$4n^2\int_M g(H, H) = \sum_{k\geqq 1}\lambda_k^2 a_k,$$

$$4n^2\int_M g(H, \Delta H) = \sum_{k\geqq 1}\lambda_k^3 a_k,$$

$$4n^2\int_M g(\Delta H, \Delta H) = \sum_{k\geqq 1}\lambda_k^4 a_k.$$

We put

$$\Phi_1 = 4n^2\int_M g(H, H)+2n\lambda_1\int_M g(X, H),$$

$$\Phi_2 = 4n^2\int_M g(H, \Delta H)-4n^2\lambda_1\int_M g(H, H),$$

$$\Phi_3 = 4n^2\int_M g(\Delta H, \Delta H)-4n^2\lambda_1\int_M g(H, \Delta H),$$

$$\Phi_4 = 4n^2\int_M g(\Delta H, \Delta H)-4n^2(\lambda_1+\lambda_2)\int_M g(H, \Delta H)+4n^2\lambda_1\lambda_2\int_M g(H, H),$$

$$\Phi_5 = 4n^2\int_M g(\Delta H, \Delta H)-4n^2(\lambda_1+\lambda_2+\lambda_3)\int_M g(H, \Delta H)$$
$$+4n^2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\int_M g(H, H)+2n\lambda_1\lambda_2\lambda_3\int_M g(X, H).$$

Then we get

(5.7) $$\Phi_1 = \sum_{k\geqq 2}\lambda_k(\lambda_k-\lambda_1)a_k \geqq 0,$$

(5.8) $$\Phi_2 = \sum_{k\geqq 2}\lambda_k^2(\lambda_k-\lambda_1)a_k \geqq 0,$$

(5.9) $$\Phi_3 = \sum_{k\geqq 2}\lambda_k^3(\lambda_k-\lambda_1)a_k \geqq 0,$$

(5.10) $$\Phi_4 = \Phi_3-\lambda_2\Phi_2 = \sum_{k\geqq 3}\lambda_k^2(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)a_k \geqq 0,$$

(5.11) $$\Phi_5 = \Phi_4-\lambda_3(\Phi_2-\lambda_2\Phi_1) = \sum_{k\geqq 4}\lambda_k(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)(\lambda_k-\lambda_3)a_k \geqq 0.$$

We put

(5.12) $$\Phi_6 = \Phi_2-\lambda_2\Phi_1 = \sum_{k\geqq 3}\lambda_k(\lambda_k-\lambda_2)(\lambda_k-\lambda_1)a_k \geqq 0.$$

The equality in (5.7) holds if and only if the immersion is of order 1, the equality in (5.12) holds if and only if the immersion is of order 1 or $\{1, 2\}$, and the equality in (5.11) holds if and only if the immersion is of order 1 or $\{1, 2\}$ or $\{1, 3\}$ or $\{2, 3\}$ or $\{1, 2, 3\}$.

Thus we have

PROPOSITION 2 (N. Ejiri, A. Ros, see [**9**]). *Let M be an n-dimensional compact Kaehler submanifold immersed in $\boldsymbol{C}P^m$. Then*

$$\lambda_1 \leqq n+1 .$$

The equality holds if and only if M is totally geodesic (that is, of order 1).

PROPOSITION 3. *Let M be an n-dimensional compact Kaehler submanifold immersed in $\boldsymbol{C}P^m$.*

If $\lambda_1=\int_M \tau \Big/ (n\,\mathrm{vol}(M))$ and M is not totally geodesic, then

$$\lambda_2 \leqq n+2 .$$

The equality holds if and only if M is Einstein and the second fundamental form of the immersion is parallel (that is, of order $\{1, 2\}$).

PROOF. In Corollary 5.4 in [**9**], under the same assumptions as Proposition 3, it is proved that $\lambda_2 \leqq n+2$ and the equality holds only if M is Einstein and the second fundamental form of the immersion is parallel. Hence it is enough to prove that if M is an Einstein parallel submanifold, then $\lambda_2=n+2$. But, from Theorem 7.4 in [**6**], all Einstein parallel submanifolds are listed in Table 1, which, together with the result obtained in § 4, shows that $\lambda_2=n+2$. Using Lemma 2 and (5.12) we see that $\lambda_2=n+2$ if and only if the equality in (5.12) holds since M is not totally geodesic. But since the equality in (5.12) holds if and only if M is of order $\{1, 2\}$, the proof of Proposition 3 is accomplished.

Table 1. Einstein Kaehler submanifolds of degree 2.

submanifold	$\dim_C$	p	τ	λ_1	λ_2
$M_1=\boldsymbol{C}P^n(1/2)$	n	$n(n+1)/2$	$n(n+1)/2$	$(n+1)/2$	$n+2$
$M_2=\boldsymbol{Q}^n$	n	1	n^2	n	$n+2$
$M_3=\boldsymbol{C}P^n \times \boldsymbol{C}P^n$	$2n$	n^2	$2n(n+1)$	$n+1$	$2n+2$
$M_4=U(s+2)/U(2)\times U(s)$ $(s\geqq 3)$	$2s$	$s(s-1)/2$	$2s(s+2)$	$s+2$	$2s+2$
$M_5=SO(10)/U(5)$	10	5	80	8	12
$M_6=E_6/Spin(10)\times T$	16	10	192	12	18

Since dimension, vol(M), and $\int_M \tau$ are spectral invariants, from Proposition 3 and Table 1, we have

PROPOSITION 4. *Let M be an n-dimensional compact Kaehler submanifold immersed in CP^m. If* Spec(M)=Spec(M_i) *for some $i=1, \cdots 6$, then M is congruent to the standard imbedding of M_i, where M_i is one of the Hermitian symmetric spaces given in Table* 1.

REMARK. Proposition 4 for $i=1, \cdots, 5$ is obtained in [9].

The following formulas are well-known (for example, see [7], [9]),

(5.13) $$\tau = n(n+1) - \|\sigma\|^2,$$

(5.14) $$\|S\|^2 = \frac{1}{2} n(n+1)^2 - (n+1)\|\sigma\|^2 + \mathrm{tr}(\sum_\lambda A_\lambda^2)^2,$$

(5.15) $$\|R\|^2 = 2n(n+1) - 4\|\sigma\|^2 + 2\|T\|^2,$$

(5.16) $$-\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla\sigma\|^2 + \frac{n+2}{2}\|\sigma\|^2 - 2\,\mathrm{tr}(\sum_\lambda A_\lambda^2)^2 - \|\mathrm{T}\|^2,$$

(5.17) $$\frac{n(n+1)}{2}\|R\|^2 \geqq 2n\|S\|^2 \geqq \tau^2.$$

The first equality in (5.17) holds if and only if M has constant holomorphic sectional curvature, and the second equality in (5.17) holds if and only if M is Einstein.

From (5.13), (5.14) and (5.17), we have

(5.18) $$\mathrm{tr}(\sum_\lambda A_\lambda^2)^2 \geqq \frac{1}{2n}\|\sigma\|^4.$$

The equality holds if and only if M is Einstein.

LEMMA 3. *Let M be an n-dimensional compact Kaehler submanifold immersed in CP^m with the following properties:*

i) $$\lambda_1 = \frac{\int_M \tau}{n\,\mathrm{vol}(M)},$$

ii) $$\lambda_2 = \frac{(n+3)\lambda_1 - \int_M (\|R\|^2 + 2\|S\|^2)\big/(n\,\mathrm{vol}(M))}{n+1-\lambda_1} + \lambda_1,$$

iii) $\nabla\sigma \neq 0$.

Then

$$\lambda_3 \leqq n+3.$$

The equality holds only if the immersion is of order $\{1, 3\}$ *or* $\{2, 3\}$ *or* $\{1, 2, 3\}$. *Moreover, λ_1, λ_2, λ_3 and $\|\sigma\|^2$ are given as follows: For the case of order* $\{1, 3\}$,

$$\begin{aligned}\lambda_1 &= \frac{n(n+p+1)-p}{n+2p}, \qquad \lambda_2 = n+1, \qquad \lambda_3 = n+3, \\ \|\sigma\|^2 &= \frac{np(n+3)}{n+2p},\end{aligned} \tag{5.19}$$

and, for the case of order $\{2, 3\}$,

$$\begin{aligned}\lambda_1 &= \frac{2n(n+1)+p(n-3)}{2n+3p}, \\ \lambda_2 &= \frac{2n(n+p+1)}{2n+3p}, \qquad \lambda_3 = n+3, \\ \|\sigma\|^2 &= \frac{2np(n+3)}{2n+3p},\end{aligned} \tag{5.20}$$

where p *is the full codimension.*

PROOF. Using Lemma 2, (5.13), (5.14) and (5.15), we have

$$\begin{aligned}\Phi_5 = 2n\,\mathrm{vol}(M)\{&(n+1)(n+2)(n+3)-(n+1)(n+2)(\lambda_1+\lambda_2+\lambda_3) \\ &+(n+1)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)-\lambda_1\lambda_2\lambda_3\} \\ &+2(\lambda_1+\lambda_2+\lambda_3-4n-8)\int_M \tau+2\int_M(\|R\|^2+2\|S\|^2).\end{aligned}$$

From the assumptions i) and ii), we get

$$\Phi_5 = 2n\,\mathrm{vol}(M)(n+1-\lambda_1)(n+2-\lambda_2)(n+3-\lambda_3).$$

From the assumption iii), Proposition 2, Proposition 3 and (5.11), we have

$$\lambda_3 \leqq n+3.$$

If the immersion is of order $\{k_1, k_2\}$, then the following holds (see [9]):

$$\lambda_{k_1}+\lambda_{k_2} = n+1+\frac{(n+p)\|\sigma\|^2}{np}. \tag{5.21}$$

It follows from Proposition 1 that M is an Einstein Kaehler submanifold with $T=kg$. Hence we obtain

$$\lambda_1 = \frac{\tau}{n} = \frac{n(n+1)-\|\sigma\|^2}{n}. \tag{5.22}$$

Since $\|T\|^2=\|\sigma\|^4/2p$ (see [9]), from (5.14), (5.15) and (5.18), we have

$$\frac{\int_M(\|R\|^2+2\|S\|^2)}{n\,\mathrm{vol}(M)} = (n+1)(n+3)-\frac{2(n+3)\|\sigma\|^2}{n}+\frac{(n+p)\|\sigma\|^4}{n^2p}. \tag{5.23}$$

From (5.21), (5.22), (5.23) and $\lambda_3=n+3$, we have (5.19) for the case of order $\{1, 3\}$, and (5.20) for the case of order $\{2, 3\}$. Q. E. D.

§ 6. Spectral geometry for Kaehler submanifolds II.

In this section, we investigate the order of $CP^n(1/3)$ and compact irreducible Hermitian symmetric spaces of rank 3. We choose a local field of unitary frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ on CP^{n+p} in such a way that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. With respect to the frame field on CP^{n+p}, let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+p}\}$ be the field of dual frames. Then the Kaehler metric of CP^{n+p} is given by $\sum_{A=1}^{n+p}\omega^A\cdot\bar{\omega}^A$ and the structure equations of CP^{n+p} are given by

$$d\omega^A+\sum_B\omega_B^A\wedge\omega^B=0, \qquad \omega_B^A+\bar{\omega}_A^B=0, \tag{6.1}$$

$$d\omega_B^A+\sum_C\omega_C^A\wedge\omega_B^C=\tilde{\Omega}_B^A, \qquad \tilde{\Omega}_B^A=\sum_{C,D}R_{BC\bar{D}}^A\omega^C\wedge\bar{\omega}^D. \tag{6.2}$$

Since CP^{n+p} is a complex space form of constant holomorphic sectional curvature 1, we have

$$\tilde{R}_{BC\bar{D}}^A=\frac{1}{4}(\delta_B^A\delta_{CD}+\delta_C^A\delta_{BD}). \tag{6.3}$$

Restricting these forms to M^n, we have

$$\omega^\alpha=0, \tag{6.4}$$

and the Kaehler metric g of M^n is given by $g=\sum_a\omega^a\cdot\bar{\omega}^a$. Moreover we obtain

$$\omega_a^\alpha=\sum_b k_{ab}^\alpha\omega^b, \qquad k_{ab}^\alpha=k_{ba}^\alpha, \tag{6.5}$$

$$d\omega^a+\sum_b\omega_b^a\wedge\omega^b=0, \qquad \omega_b^a+\bar{\omega}_a^b=0, \tag{6.6}$$

$$d\omega_b^a+\sum_c\omega_c^a\wedge\omega_b^c=\Omega_b^a, \qquad \Omega_b^a=\sum_{c,d}R_{bc\bar{d}}^a\omega^c\wedge\bar{\omega}^d, \tag{6.7}$$

$$d\omega_\beta^\alpha+\sum_\gamma\omega_\gamma^\alpha\wedge\omega_\beta^\gamma=\Omega_\beta^\alpha, \qquad \Omega_\beta^\alpha=\sum_{c,d}R_{\beta c\bar{d}}^\alpha\omega^c\wedge\bar{\omega}^d. \tag{6.8}$$

From (6.5) and (6.7), we have the equation of Gauss

$$R_{bc\bar{d}}^a=\frac{1}{4}(\delta_b^a\delta_{cd}+\delta_c^a\delta_{bd})-\sum_\alpha k_{bc}^\alpha\bar{k}_{ad}^\alpha, \tag{6.9}$$

and from (6.5), (6.6) and (6.8), we have

$$R_{\beta c\bar{d}}^\alpha=\frac{1}{4}\delta_\beta^\alpha\delta_{cd}+\sum_a k_{ac}^\beta\bar{k}_{ad}^\alpha. \tag{6.10}$$

The Ricci tensor $S_{c\bar{d}}$ and the scalar curvature τ of M^n are given by

$$S_{c\bar{d}}=\frac{n+1}{2}\delta_{cd}-2\sum_{\alpha,a}k_{ac}^\alpha\bar{k}_{ad}^\alpha, \tag{6.11}$$

$$\tau=n(n+1)-4\sum_{\alpha,c,d}k_{cd}^\alpha\bar{k}_{cd}^\alpha. \tag{6.12}$$

Now, we define the covariant derivatives k^{α}_{abc} and $k^{\alpha}_{ab\bar{c}}$ of k^{α}_{ab} by

$$\sum_c k^{\alpha}_{abc}\omega^c+\sum_c k^{\alpha}_{ab\bar{c}}\bar{\omega}^c = dk^{\alpha}_{ab}-\sum_c k^{\alpha}_{cb}\omega^c_a-\sum_c k^{\alpha}_{ac}\omega^c_b+\sum_\beta k^{\beta}_{ab}\omega^{\alpha}_{\beta}.$$

Then we have

$$k^{\alpha}_{abc}=k^{\alpha}_{bac}=k^{\alpha}_{acb}, \qquad k^{\alpha}_{ab\bar{c}}=0. \tag{6.13}$$

We can define inductively the covariant derivatives $k^{\alpha}_{a_1\cdots a_m a_{m+1}}$ and $k^{\alpha}_{a_1\cdots a_m \bar{a}_{m+1}}$ of $k^{\alpha}_{a_1\cdots a_m}$ for $m\geqq 2$. It is clear that

$$(\bar{k}^{\alpha}_{a_1\cdots a_m})_b=\bar{k}^{\alpha}_{a_1\cdots a_m \bar{b}} \quad \text{and} \quad (\bar{k}^{\alpha}_{a_1\cdots a_m})_{\bar{b}}=\bar{k}^{\alpha}_{a_1\cdots a_m b}.$$

We see that $k^{\alpha}_{a_1\cdots a_m}$ is symmetric with respect to $a_1, \cdots, a_m$. The following formula is proved in [**6**]:

LEMMA 4.

$$\begin{aligned} k^{\alpha}_{a_1\cdots a_m\bar{b}} &= \frac{m-2}{4}\sum_{r=1}^{m} k^{\alpha}_{a_1\cdots\hat{a}_r\cdots a_m}\delta_{a_r b} \\ &\quad -\sum_{r=1}^{m-2}\frac{1}{r!(m-r)!}\sum_{\sigma,\beta,c} k^{\alpha}_{ca_{\sigma(1)}\cdots a_{\sigma(r)}}k^{\beta}_{a_{\sigma(r+1)}\cdots a_{\sigma(m)}}\bar{k}^{\beta}_{cb} \end{aligned} \tag{6.14}$$

for $m\geqq 3$, *where the summation on* σ *is taken over all permutations of* $(1, \cdots, m)$.

Let $T_x(M)$ be the tangent space to M at x and $T^{C}_x(M)$ its complexification. Let $T^{1,0}_x(M)=\{X-\sqrt{-1}JX \mid X\in T_x(M)\}$ and $T^{0,1}_x(M)=\{X+\sqrt{-1}JX \mid X\in T_x(M)\}$. Then

$$T^{C}_x(M)=T^{1,0}_x(M)+T^{0,1}_x(M).$$

The similar results hold for CP^{n+p}. Suppose that the relation between e_A and E_A is given by

$$e_A=\frac{1}{2}(E_A-\sqrt{-1}\,E_{A^*}), \qquad e_{\bar{A}}=\frac{1}{2}(E_A+\sqrt{-1}\,E_{A^*}).$$

Then, the relation between h^{α}_{ab} and k^{α}_{ab} is given by (see [**7**])

$$\begin{aligned} k^{\alpha}_{ab} &= h^{\alpha}_{ab}-\sqrt{-1}\,h^{\alpha}_{ab^*}, \\ \bar{k}^{\alpha}_{ab} &= h^{\alpha}_{ab}+\sqrt{-1}\,h^{\alpha}_{ab^*}. \end{aligned} \tag{6.15}$$

Moreover we can see that

$$\begin{aligned} k^{\alpha}_{abc} &= h^{\alpha}_{abc}-\sqrt{-1}\,h^{\alpha}_{ab^*c}, \\ \bar{k}^{\alpha}_{abc} &= h^{\alpha}_{abc}+\sqrt{-1}\,h^{\alpha}_{ab^*c}. \end{aligned} \tag{6.16}$$

Thus we have

$$\|\sigma\|^2 = \sum_{\lambda, i, j} h^\lambda_{ij} h^\lambda_{ij} = 4 \sum_{\alpha, a, b} k^\alpha_{ab} \bar{k}^\alpha_{ab},$$

$$\|T\|^2 = \sum_{\lambda, \mu, i, j, k, l} h^\lambda_{ij} h^\mu_{ij} h^\mu_{kl} h^\lambda_{kl} = 8 \sum_{\alpha, \beta, a, b, c, d} k^\alpha_{ab} \bar{k}^\beta_{ab} k^\beta_{cd} \bar{k}^\alpha_{cd}, \tag{6.17}$$

$$\|\nabla\sigma\|^2 = \sum_{\lambda, i, j, k} h^\lambda_{ijk} h^\lambda_{ijk} = 8 \sum_{\alpha, a, b, c} k^\alpha_{abc} \bar{k}^\alpha_{abc}.$$

The Laplacian is given by

$$\Delta = -4 \sum_a \nabla_{\bar{a}} \nabla_a .$$

We define A_m by

$$A_m = \sum_{\alpha, a_1, \cdots, a_m} k^\alpha_{a_1 \cdots a_m} \bar{k}^\alpha_{a_1 \cdots a_m}.$$

Now, we say that the immersion is of *degree* m_0 if there exists a positive integer m_0 in such a way that $A_{m_0} \neq 0$, $A_{m_0+1} = 0$. We need the following.

LEMMA 5 ([**11**]). *Let $f_{p_i}: M_i \rightarrow \boldsymbol{CP}^m$ be the p_i-th full Kaehler imbedding of a compact irreducible Hermitian symmetric space M_i of rank r_i, and let f be the tensor product of f_{p_i} $(i=1, \cdots, s)$. Then the degree of f is $\sum_{i=1}^s p_i r_i$.*

If M is an n-dimensional locally symmetric Einstein Kaehler submanifold with $T = kg$ (see, Proposition 1), then we have

$$\sum_\alpha k^\alpha_{abc} \bar{k}^\alpha_{de} = 0 \qquad \text{and} \qquad \sum_{a, b} k^\alpha_{abc} \bar{k}^\beta_{ab} = 0,$$

so that from (6.14) we get

$$\sum_d k^\alpha_{abcd\bar{d}} = \left(\frac{n+3}{2} - \frac{3\|\sigma\|^2}{4n} \right) k^\alpha_{abc}.$$

Hence if M is an n-dimensional locally symmetric Einstein Kaehler submanifold with $T = kg$, $A_4 = 0$ and $\tau \neq n(n-3)/3$, then $\nabla\sigma = 0$. Therefore, from Proposition 1, Lemma 5 and Table 2, we see that $\boldsymbol{CP}^n(1/3)$ and compact irreducible Hermitian symmetric spaces of rank 3 cannot be of order $\{k_1, k_2\}$. Consequently, from Lemma 3 and Table 2 we have the following.

PROPOSITION 5. *Compact irreducible Hermitian symmetric submanifolds of degree* 3 *are of order* $\{1, 2, 3\}$.

§7. Proof of Theorem.

Let R, S, τ, T, σ be the curvature tensor, the Ricci tensor, the scalar curvature, the tensor given in (3.5) and the second fundamental form of M respectively, and let $\tilde{R}$, $\tilde{S}$, $\tilde{\tau}$, $\tilde{T}$ and $\tilde{\sigma}$ be the ones of $\tilde{M}$. First, we get (see [**1**])

$$\dim(M) = \dim(\tilde{M}), \qquad \operatorname{vol}(M) = \operatorname{vol}(\tilde{M}), \qquad \int_M \tau = \int_{\tilde{M}} \tilde{\tau}$$

and

$$\int_M (2\|R\|^2-2\|S\|^2+5\tau^2)=\int_{\tilde{M}}(2\|\tilde{R}\|^2-2\|\tilde{S}\|^2+5\tilde{\tau}^2).$$

These, together with the fact that M and $\tilde{M}$ are Einstein, yield

$$\tau=\tilde{\tau},\qquad \|S\|^2=\|\tilde{S}\|^2 \quad\text{and}\quad \int_M \|R\|^2=\int_{\tilde{M}}\|\tilde{R}\|^2.$$

Then, from (5.13), (5.15) and (5.16) we see that

$$\|\sigma\|^2=\|\tilde{\sigma}\|^2,\qquad \int_M \|T\|^2=\int_{\tilde{M}}\|\tilde{T}\|^2 \quad\text{and}\quad \int_M \|\nabla\sigma\|^2=\int_{\tilde{M}}\|\tilde{\nabla}\tilde{\sigma}\|^2. \tag{7.1}$$

Moreover, since M is Einstein,

$$\int_M \left(-\frac{1}{9}\|\nabla R\|^2+\frac{8}{21}\sum R^*_{ijkl}R^*_{klmn}R^*_{mnij}\right) \tag{7.2}$$

is a spectral invariant (see [**10**]), where R^*_{ijkl} denotes the components of R with respect to the real local orthonormal frames.

We see that

$$\sum R^*_{ijkl}R^*_{klmn}R^*_{mnij}=64\sum R^a_{bc\bar{d}}R^c_{de\bar{f}}R^e_{fa\bar{b}}. \tag{7.3}$$

From (6.9) we get

$$\sum R^a_{bc\bar{d}}R^c_{de\bar{f}}R^e_{fa\bar{b}}=\frac{n(n+1)(n+3)}{64}-\frac{(n+3)\|\sigma\|^2}{32}+\frac{\|\sigma\|^4}{32n}+\frac{\|T\|^2}{16}$$
$$-\sum k^\alpha_{ab}\bar{k}^\beta_{bc}k^\gamma_{cd}\bar{k}^\alpha_{de}k^\beta_{ef}\bar{k}^\gamma_{fa}.$$

This, together with (7.1)$\sim$(7.3), implies that

$$\int_M \left(-\frac{1}{9}\|\nabla R\|^2-\frac{512}{21}\sum k^\alpha_{ab}\bar{k}^\beta_{bc}k^\gamma_{cd}\bar{k}^\alpha_{de}k^\beta_{ef}\bar{k}^\gamma_{fa}\right) \tag{7.4}$$

is a spectral invariant. From Lemma 4, we have

$$k^\alpha_{abc\bar{d}}=\frac{1}{4}(k^\alpha_{bc}\delta_{ad}+k^\alpha_{ac}\delta_{bd}+k^\alpha_{ab}\delta_{cd})$$
$$-\sum_{\beta,e}(k^\alpha_{ea}k^\beta_{bc}+k^\alpha_{eb}k^\beta_{ca}+k^\alpha_{ec}k^\beta_{ab})\bar{k}^\beta_{ed},$$

from which it follows that

$$\sum_{\alpha,a,b,c}k^\alpha_{abc}(\bar{k}^\alpha_{abc})_d=\frac{3}{4}\sum k^\alpha_{abd}\bar{k}^\alpha_{ab}-3\sum k^\alpha_{abc}\bar{k}^\alpha_{ae}k^\beta_{ed}\bar{k}^\beta_{bc}.$$

Since M is Einstein, we have

$$\sum_{\alpha,a}k^\alpha_{abc}\bar{k}^\alpha_{ae}=(\sum_{\alpha,a}k^\alpha_{ab}\bar{k}^\alpha_{ae})_c=0,$$

so that we get

$$\sum_{\alpha,a,b,c}k^\alpha_{abc}(\bar{k}^\alpha_{abc})_d=0. \tag{7.5}$$

Then it follows that

$$-\frac{1}{4}\Delta A_3 = \sum_{\alpha,a,b,c,d}(k^{\alpha}_{abc}\bar{k}^{\alpha}_{abc})_{d\bar{d}}$$
$$= \sum k^{\alpha}_{abcd\bar{d}}\bar{k}^{\alpha}_{abc} + \sum k^{\alpha}_{abcd}\bar{k}^{\alpha}_{abcd}$$
$$= \sum k^{\alpha}_{abcd\bar{d}}\bar{k}^{\alpha}_{abc} + A_4 .$$

Hence we obtain

$$\int_M A_4 = -\int_M \sum k^{\alpha}_{abcd\bar{d}}\bar{k}^{\alpha}_{abc}.$$

Then, from Lemma 4, we see that

$$\int_M A_4 = \int_M \{-[(n+3)/2 - 3\|\sigma\|^2/4n]A_3 + \sum k^{\alpha}_{abc}\bar{k}^{\alpha}_{ade}k^{\beta}_{de}\bar{k}^{\beta}_{abc} + 3\sum k^{\alpha}_{abc}\bar{k}^{\beta}_{ab}k^{\beta}_{de}\bar{k}^{\alpha}_{dec}\} . \tag{7.6}$$

On the other hand, from (7.5) we get

$$0 = \sum\{k^{\alpha}_{abc}(\bar{k}^{\alpha}_{abc})_d\}_{\bar{d}} = \sum k^{\alpha}_{abc\bar{d}}(\bar{k}^{\alpha}_{abc})_d + \sum k^{\alpha}_{abc}(\bar{k}^{\alpha}_{abc})_{d\bar{d}}$$
$$= \frac{3(n+2)\|\sigma\|^2}{64} - \frac{3}{16}\Big(\|T\|^2 + \frac{\|\sigma\|^4}{n}\Big) + \frac{3\|\sigma\|^2\|T\|^2}{32n}$$
$$+\frac{3}{4}A_3 - 3\sum k^{\alpha}_{abc}\bar{k}^{\beta}_{ab}\bar{k}^{\alpha}_{edc}k^{\beta}_{ed} + 6\sum k^{\alpha}_{ab}\bar{k}^{\beta}_{bc}k^{\gamma}_{cd}\bar{k}^{\alpha}_{de}k^{\beta}_{ef}\bar{k}^{\gamma}_{fa} ,$$

from which it follows that

$$\sum k^{\alpha}_{ab}\bar{k}^{\beta}_{bc}k^{\gamma}_{cd}\bar{k}^{\alpha}_{de}k^{\beta}_{ef}\bar{k}^{\gamma}_{fa} = \frac{1}{2}\sum k^{\alpha}_{abc}\bar{k}^{\beta}_{ab}\bar{k}^{\alpha}_{edc}k^{\beta}_{ed} + \text{term of } \{n,\ \|\sigma\|^2,\ \|T\|^2,\ A_3\}. \tag{7.7}$$

This, together with (7.6), implies

$$\int_M \sum k^{\alpha}_{ab}\bar{k}^{\beta}_{bc}k^{\gamma}_{cd}\bar{k}^{\alpha}_{de}k^{\beta}_{ef}\bar{k}^{\gamma}_{fa} = \int_M \Big(\frac{1}{6}A_4 - \frac{1}{6}\sum k^{\alpha}_{abc}\bar{k}^{\alpha}_{de}k^{\beta}_{de}\bar{k}^{\beta}_{abc} + \text{term of } \{n,\ \|\sigma\|^2,\ \|T\|^2,\ A_3\}\Big). \tag{7.8}$$

Therefore, from (7.4) and (7.8) we see that

$$\int_M \Big(-\frac{1}{9}\|\nabla R\|^2 + \frac{256}{63}\sum k^{\alpha}_{abc}\bar{k}^{\alpha}_{de}k^{\beta}_{de}\bar{k}^{\beta}_{abc} - \frac{256}{63}A_4\Big)$$

is a spectral invariant. On the other hand, from (6.15) and (6.16) we get

$$\|\nabla R\|^2 = 64\sum k^{\alpha}_{abc}\bar{k}^{\alpha}_{de}k^{\beta}_{de}\bar{k}^{\beta}_{abc},$$

from which it follows that

Table 2. Compact irreducible Hermitian symmetric submanifolds of degree 3.

submanifold	$\dim_C$	p	$\|S\|^2$	$\|R\|^2$	τ	$\|\sigma\|^2$	$\|T\|^2$	μ	λ_1	λ_2	λ_3
$\boldsymbol{CP}^n(1/3)$	n	$n(n+1)(n+5)/6$	$n(n+1)^2/18$	$2n(n+1)/9$	$n(n+1)/3$	$2n(n+1)/3$	$4n(n+1)/9$	$1/3$	$(n+1)/3$	$2(n+2)/3$	$n+3$
$SU(r+3)/S(U(r)\times U(3))$ $(r\geqq 3)$	$3r$	$r(r-1)(r+7)/6$	$3r(r+3)^2/2$	$6r(3r+1)$	$3r(r+3)$	$6r(r-1)$	$12r(r-1)$	-1	$r+3$	$2r+4$	$3r+3$
$Sp(3)/U(3)$	6	7	48	66	24	18	27	$-1/2$	4	7	9
$SO(12)/U(6)$	15	16	750	660	150	90	270	-2	10	16	18
$SO(14)/U(7)$	21	42	1512	1344	252	210	630	-2	12	20	24
$E_7/E_6\times T$	27	28	4374	3132	486	270	1350	-4	18	28	30

Table 3. Compact reducible Einstein Hermitian symmetric submanifolds of degree 3.

submanifold	$\dim_C$	λ_1	λ_2	λ_3
$\boldsymbol{CP}^n\times\boldsymbol{CP}^n\times\boldsymbol{CP}^n$	$3n$	$n+1$	$2n+2$	$2n+4$
$\boldsymbol{CP}^n\times\boldsymbol{CP}^{2n+1}(1/2)$	$3n+1$	$n+1$	$2n+2$	$2n+3$
$\boldsymbol{CP}^n\times\boldsymbol{Q}^{n+1}$ $(n\geqq 2)$	$2n+1$	$n+1$	$n+3$	$2n+2$
$\boldsymbol{CP}^n\times\{SU(n+1)/S(U(2)\times U(n-1))\}$ $(n\geqq 4)$	$3n-2$	$n+1$	$2n$	$2n+2$
$\boldsymbol{CP}^7\times\{SO(10)/U(5)\}$	17	8	12	16
$\boldsymbol{CP}^{11}\times\{E_6/Spin(10)\times T\}$	27	12	18	24

$$\int_M (3\|\nabla R\|^2+256A_4)$$

is a spectral invariant. Since $\tilde{M}$ is locally symmetric and of degree 3, it follows that M is also locally symmetric and of degree $\leqq 3$. Hence M is a compact Hermitian symmetric submanifold of degree $\leqq 3$. From Lemma 5, Proposition 2, Tables 1~3 and Theorem 4.3 in [**6**], M is one of the compact Hermitian symmetric submanifolds given in Tables 2 and 3. Q. E. D.

Eigenvalues for classical symmetric spaces (up to their ranks) are computed by T. Nagano [**5**] and eigenvalues for exceptional types are computed in § 4, and eigenvalues for ones given in Table 3 can be computed in the same way. And from Lemma 2.4 in [**6**], we get

$$\|T\|^2=(1-\mu)\|\sigma\|^2,$$

where μ is given in Table 2. Since the scalar curvatures for irreducible Hermitian symmetric spaces are given in Table 2 of [**6**], from the above formula and (5.13)~(5.16) we can compute the values of $\|\sigma\|^2$, $\|T\|^2$, $\|S\|^2$ and $\|R\|^2$.

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