TOKYO J. MATH.
VOL. 10, NO. 1, 1987

Minimal Immersions of Kaehler Manifolds into Complex Space Forms

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(Communicated by K. Ogiue)

Introduction

Let $N^m(\tilde{c})$ be an m-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$, and let (M^n, g) be an n-dimensional Kaehler manifold. It is well known that holomorphic isometric immersions of Kaehler manifolds into Kaehler manifolds are minimal immersions. We consider the following problem: Is an isometric minimal immersion $f: (M^n, g)\to N^m(\tilde{c})$ a holomorphic or anti-holomorphic immersion? However, it is not true in general. For example, if we take $M^1=\boldsymbol{R}H^2(\tilde{c}/4)$ (2-dimensional real hyperbolic space of constant curvature $\tilde{c}/4$, $\tilde{c}<0$) or $M^1=S^2(\tilde{c}/4)$ (2-dimensional sphere of constant curvature $\tilde{c}/4$, $\tilde{c}>0$), then we obtain totally real isometric minimal immersions as follows:

$$(1)\qquad M^1=\boldsymbol{R}H^2(\tilde{c}/4)\xrightarrow{\text{totally real, totally geodesic}}\boldsymbol{C}H^m(\tilde{c})\ ,$$

where $\boldsymbol{C}H^m(\tilde{c})$ is an m-dimensional complex hyperbolic space of constant holomorphic sectional curvature $\tilde{c}$.

$$(2)\qquad M^1=S^2(\tilde{c}/4)\xrightarrow{\text{natural covering}}\boldsymbol{R}P^2(\tilde{c}/4)\xrightarrow{\text{totally real, totally geodesic}}\boldsymbol{C}P^m(\tilde{c})\ ,$$

where $\boldsymbol{C}P^m(\tilde{c})$ is an m-dimensional complex projective space of constant holomorphic sectional curvature $\tilde{c}$ and $\boldsymbol{R}P^2(\tilde{c}/4)$ is a 2-dimensional real projective space of constant curvature $\tilde{c}/4$.

In Part I, we prove the following

THEOREM 1. *Let $\boldsymbol{C}H^m(\tilde{c})$ be an m-dimensional complex hyperbolic space of constant holomorphic sectional curvature $\tilde{c}$ ($\tilde{c}<0$), and let (M^n, g) be an n-dimensional Kaehler manifold such that* $\dim_{\boldsymbol{C}} M=n\geqq 2$. *Then,*

Received September 6, 1986

evey minimal isometric immersion of (M^n, g) *into* $CH^m(\tilde{c})$ *is a holomorphic or anti-holomorphic immersion.*

REMARK 1. By (1), Theorem 1 is not true if $\dim_C M=1$. Our result is closely related to Siu's work [5]. In fact, by using Siu's arguments we can show that Theorem 1 holds even if $CH^m(\tilde{c})$ is replaced by a Kaehler manifold of strongly negative curvature tensor. On the other hand, we can prove Theorem 1 by using Gauss equation only. $CH^m(\tilde{c})$ and its quotient manifolds are the examples of Kaehler manifolds of strongly negative curvature tensor (for example, see [5], [6]). Since Theorem 1 is of local nature, we can easily see that Theorem 1 holds when $CH^m(\tilde{c})$ is replaced by its quotient manifolds.

In Part II, we prove the following theorem, which is some generalization of Micallef's result [4].

THEOREM 2. *Let* $f: M^n \to \boldsymbol{R}^{2n+2}/D$ *be an isometric stable minimal immersion of an n-dimensional compact Kaehler manifold into a* $(2n+2)$*-dimensional flat torus. Assume that* $|R|^2 \geqq \tau^2$ *holds on M, where R is the curvature tensor and* τ *is the scalar curvature of M. Then, f is holomorphic with respect to some orthogonal complex structure of* $\boldsymbol{R}^{2n+2}/D$.

REMARK 2. The case of $n=1$ in Theorem 2 is proved by Micallef ([4], p. 73, Theorem 1′), because the condition $|R|^2 \geqq \tau^2$ is automatically satisfied when $n=1$. For every complex hypersurface in $\boldsymbol{C}^{n+1}/D$, $|R|^2=\tau^2$ holds.

The author wishes to thank Professors K. Ogiue, N. Ejiri and H. Tsuji for many valuable comments and suggestions.

PART I. THE CASE OF $\tilde{c}<0$

§1. Proof of Theorem 1.

Let $f: (M^n, g) \to (N^m(\tilde{c}), h)$ be an isometric minimal immersion of an n-dimensional Kaehler manifold into an m-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. Let TM^C (resp. TN^C) be the complexification of the tangent bundle of M (resp. N). Then, the differential mapping $f_*: TM \to TN$ can be naturally extended to the complex linear mapping, which is also denoted by f_*. We have the Gauss equation

$$(3) \qquad h(\tilde{R}(f_*(X), f_*(Y))f_*(Z), f_*(W))$$

$$=g(R(X, Y)Z, W)+h(\sigma(X, Z), \sigma(Y, W))$$
$$-h(\sigma(Y, Z), \sigma(X, W)) , \quad \text{for} \quad X, Y, Z, W\in TM^c ,$$

where R, $\tilde{R}$ and σ denote the curvature tensor of M, the curvature tensor of N and the second fundamental form of f, respectively. They are also extended to the complex tensors. To prove Theorem 1, it is enough to show that f is a holomorphic or anti-holomorphic immersion at each point $p\in M$. Therefore, we choose any point $p\in M$ and we verify the assertion at p. We choose unitary bases $e_1, \cdots, e_n$ of T_pM^c and $u_1, \cdots, u_m$ of $T_{f(p)}N^c$. We use the following convention on the range of indices:

$$\alpha, \beta, \gamma, \cdots=1, \cdots, m \qquad i, j, k, \cdots=1, \cdots, n$$
$$\lambda, \mu, \nu, \cdots=1, \cdots, m, \bar{1}, \cdots, \bar{m}$$
$$A, B, C, \cdots=1, \cdots, n, \bar{1}, \cdots, \bar{n} .$$

Then, we have

$$h(\tilde{R}(u_\alpha, u_{\bar{\beta}})u_\gamma, u_{\bar{\delta}})=\frac{\tilde{c}}{2}(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) . \tag{4}$$

If we put

$$f_*(e_i)=\sum_\alpha f_i^\alpha u_\alpha+\sum_\alpha f_i^{\bar{\alpha}}u_{\bar{\alpha}} ,$$

$$f_*(e_{\bar{i}})=\sum_\alpha f_{\bar{i}}^\alpha u_\alpha+\sum_\alpha f_{\bar{i}}^{\bar{\alpha}}u_{\bar{\alpha}} ,$$

then we have

$$\sum_\alpha f_i^\alpha f_{\bar{j}}^{\bar{\alpha}}+\sum_\alpha f_i^{\bar{\alpha}}f_{\bar{j}}^\alpha=\delta_{ij} \tag{5}$$

$$\sum_\alpha f_i^\alpha f_j^{\bar{\alpha}}+\sum_\alpha f_i^{\bar{\alpha}}f_j^\alpha=0 \qquad \text{for any } i \text{ and } j , \tag{6}$$

because both g and h are Kaehler metrics. Moreover, since M and N are Kaehler manifolds, we have

$$\begin{aligned} g(R(e_A, e_B)e_i, e_j)&=g(R(e_A, e_B)e_{\bar{i}}, e_{\bar{j}}) \\ &=g(R(e_i, e_j)e_A, e_B)=g(R(e_{\bar{i}}, e_{\bar{j}})e_A, e_B)=0 , \end{aligned} \tag{7}$$

$$\begin{aligned} h(\tilde{R}(u_\lambda, u_\mu)u_\alpha, u_\beta)&=h(\tilde{R}(u_\lambda, u_\mu)u_{\bar{\alpha}}, u_{\bar{\beta}}) \\ &=h(\tilde{R}(u_\alpha, u_\beta)u_\lambda, u_\mu)=h(\tilde{R}(u_{\bar{\alpha}}, u_{\bar{\beta}})u_\lambda, u_\mu)=0 , \end{aligned}$$
$$\text{for any } A, B, i, j, \lambda, \mu, \alpha \text{ and } \beta .$$

Therefore, if we put $X=e_i$, $Y=e_j$, $Z=e_{\bar{j}}$ and $W=e_{\bar{i}}$ in (3), it follows from (4), (6) and (7) that

$$(8)\qquad 2\tilde{c}\sum_{\alpha,\beta,i,j} f_i^{\alpha} f_{\bar j}^{\bar\alpha} f_{\bar i}^{\bar\beta} f_{\bar j}^{\beta} + \tilde{c}\sum_{\alpha,\beta,i,j}\left(f_i^{\alpha} f_{\bar i}^{\bar\alpha} f_{\bar j}^{\beta} f_j^{\bar\beta} - f_i^{\alpha} f_{\bar j}^{\bar\alpha} f_{\bar i}^{\beta} f_j^{\bar\beta}\right)$$
$$= \sum_{i,j} h(\sigma(e_i, e_{\bar j}), \sigma(e_{\bar i}, e_j)) - \sum_{i,j} h(\sigma(e_i, e_{\bar i})\cdot\sigma(e_j, e_{\bar j}))\ .$$

Since f is minimal, we have $\sum_i \sigma(e_i, e_{\bar i})=0$, which implies that the right hand side of (8) is non-negative, and so is the left hand side of (8). We claim that

$$(9)\qquad \sum f_i^{\alpha} f_{\bar i}^{\bar\alpha} f_{\bar j}^{\beta} f_j^{\bar\beta} \geqq \sum f_i^{\alpha} f_{\bar j}^{\bar\alpha} f_{\bar i}^{\beta} f_j^{\bar\beta}\ .$$

In fact, since $\sum_\alpha f_{\bar i}^{\alpha} f_{\bar j}^{\bar\alpha}$ is a Hermitian matrix, by choosing a suitable unitary basis, we can assume that

$$(10)\qquad \sum_\alpha f_{\bar i}^{\alpha} f_{\bar j}^{\bar\alpha} = \begin{cases} \sum_\alpha f_{\bar i}^{\alpha} f_{\bar i}^{\bar\alpha}\ (\geqq 0) & \text{if}\quad i=j \\ 0 & \text{if}\quad i\neq j\ , \end{cases}$$

which, by (5), implies that

$$(11)\qquad \sum_\alpha f_{\bar i}^{\bar\alpha} f_j^{\alpha} = \begin{cases} \sum_\alpha f_{\bar i}^{\bar\alpha} f_i^{\alpha}\ (\geqq 0) & \text{if}\quad i=j \\ 0 & \text{if}\quad i\neq j\ . \end{cases}$$

Then, the claim (9) is proved. If $\tilde{c}<0$, the first term of the left hand side of (8) is non-positive and the second term is also non-positive by (9). On the other hand, since the right hand side of (8) is non-negative, it follows that the equality in (9) holds. If we put $\lambda_i=\sum_\alpha f_{\bar i}^{\alpha} f_{\bar i}^{\bar\alpha}$ and $\mu_i=\sum_\alpha f_{\bar i}^{\bar\alpha} f_i^{\alpha}$, then from (9), (10) and (11) we have $\lambda_i\mu_j=0$ for $1\leqq i\neq j\leqq n$. By $n\geqq 2$ and (5) we obtain $\lambda_i=0$ for any i or $\mu_i=0$ for any i, which implies that f is holomorphic or anti-holomorphic. Therefore, Theorem 1 is proved.

§2. Some remarks on Section 1.

If $\tilde{c}=0$, from (8) we obtain

$$(12)\qquad \sigma(e_i, e_{\bar j})=0 \quad \text{for any } i \text{ and } j\ ,$$

which is equivalent to

$$(13)\qquad f_{i\bar j}^{\alpha}=0 \quad \text{for any } \alpha,\ i \text{ and } j\ .$$

It is well known that an isometric immersion f is minimal if and only if f is a harmonic map. The mapping f satisfying (13) is called "*pluri-*

harmonic". We remark the following

PROPOSITION 1. *Let* $f: (M^n, g) \to N^m(0)$ *be an isometric minimal immersion of a Kaehler manifold. Then,* f *is pluriharmonic.*

PROPOSITION 2. *Let* $f: (M^n, g) \to CP^N$ *be an isometric immersion of a Kaehler manifold into a complex projective space with the Fubini-Study metric. If* $n \geqq 2$ *and* f *is pluriharmonic, then* f *is holomorphic or anti-holomorphic.*

REMARK 3. Proposition 1 is already obtained in [1], p. 212, Theorem 1.2.

Moreover, we remark that a submanifold satisfying (12) is called *austere* by R. Harvey and H. B. Lawson, Jr. ([3], p. 102).

PART II. THE CASE OF $\tilde{c}=0$.

First, we state the following

(*) *Let* $f: M^n \to \boldsymbol{R}^{2n+2}$ *be an isometric stable minimal immersion of an* n*-dimensional complete Kaehler manifold into a* $(2n+2)$*-dimensional Euclidean space. Assume that* M *is parabolic, that is,* M *admits no positive non-constant superharmonic functions and that* $|R|^2 \geqq \tau^2$ *holds everywhere on* M*. Then,* f *is holomorphic with respect to some orthogonal complex structure of* $\boldsymbol{R}^{2n+2}$.

The case of $n=1$ in (*) is proved in [4]. We prove (*) by generalizing the method of [4] to Kaehler manifolds. Theorem 2 can be proved immediately by the same method as the proof of (*). Unfortunately, for the case of $n \geqq 2$, we know no examples of parabolic Kaehler manifolds.

§1. Stability condition.

Let $f: M^n \to \boldsymbol{R}^{2n+2}$ be an isometric stable minimal immersion of an n-dimensional Kaehler manifold into a $(2n+2)$-dimensional Euclidean space. We define the inner product $\langle \, , \rangle$ on $\boldsymbol{R}^{2n+2}$ by

$$\langle s, t\rangle = s \cdot \bar{t} \qquad \text{for any} \quad s, t \in \boldsymbol{C}^{2n+2}, \tag{1.1}$$

where

$$s \cdot t = \sum_{i=1}^{2n+2} s_i t_i \ \text{ for } s=(s_1, \cdots, s_{2n+2}) \text{ and } t=(t_1, \cdots, t_{2n+2}) .$$

Let TM^C be the complexification of the tangent bundle TM of M. Then, we have $TM^C = TM^{1,0} + TM^{0,1}$, where the fibre $T_pM^{1,0}$(resp. $T_pM^{0,1}$) at $p \in M$ is the $\sqrt{-1}$-eigenspace (resp. $-\sqrt{-1}$-eigenspace) of the complex structure tensor of M. Let NM be the normal bundle of M. Then, naturally NM has a complex structure defined by its orientation (i.e., rotation by 90°), and with respect to this complex structure, we have

$$NM^C = NM^{1,0} + NM^{0,1} .$$

We denote by $C_0^\infty(NM^C)$ the set of all compactly supported smooth sections of NM^C. Then, we have the stability inequality for a minimal submanifold of C^{n+1} (see [4])

$$\int_M |(ds)^T|^2 \leqq \int_M |(ds)^N|^2 \qquad \text{for any} \quad s \in C_0^\infty(NM^C) , \tag{1.2}$$

where d is the Riemannian connection of C^{n+1}, and the superscripts T and N denote orthogonal projections onto the tangent space and normal space of M respectively. Let (z^i) $(i=1, \cdots, n)$ be a local complex coordinate system on M. Then, we can write $ds = \partial s + \bar{\partial} s$, where $\partial s = \sum_i (\partial_i s) dz^i$ and $\bar{\partial} s = \sum_i (\partial_{\bar{i}} s) d\bar{z}^i$. Thus, (1.2) can be rewritten as

$$\begin{aligned} 2\int_M |(ds)^T|^2 &\leqq \int_M |ds|^2 \\ &= \int_M |\partial s|^2 + \int_M |\bar{\partial} s|^2 . \end{aligned} \tag{1.3}$$

Since the connection d is flat and s has compact support, by the integration by parts we have

$$\int_M |\partial s|^2 = \int_M |\bar{\partial} s|^2 .$$

This, together with (1.3), yields

$$\int_M |(\partial s)^T|^2 \leqq \int_M |(\bar{\partial} s)^N|^2 , \tag{1.4}$$

or

$$\int_M |(\bar{\partial} s)^T|^2 \leqq \int_M |(\partial s)^N|^2 . \tag{1.5}$$

§2. A condition for f to be holomorphic.

Micallef [4] proved the following

THEOREM A. *Let $F: M^n \to R^{2n}$ be an immersion of an n-dimensional complex manifold into an $2n$-real dimensional Euclidean space with the* *usual metric. Assume that there exist vector bundles E and V over M which satisfy the following conditions*:

i) $TM^c \simeq E \oplus \bar{E}$, $NM^c \simeq V \oplus \bar{V}$,

ii) $E \oplus V$ *is orthogonal to* $\bar{E} \oplus \bar{V}$ *with respect to* $\langle \cdot, \cdot \rangle$,

iii) $d: \Gamma(E \oplus V) \to \Gamma((E \oplus V) \otimes T^*M)$.

Then, there exist complex structures $\tilde{J}$ *and* J *on* M *and* R^{2n} *respectively such that* $\tilde{J}$ *is orthogonal with respect to the metric induced on* M *by* F, J *is orthogonal with respect to the Euclidean inner product on* R^{2n} *and* F *is holomorphic with respect to* $\tilde{J}$ *and* J.

REMARK 4. J is actually covariant constant on R^{2n}, so that R^{2n} is endowed with the usual Kaehler structure of C^n.

We can apply Theorem A to our situation. We put $E = TM^{1,0}$ and $V = NM^{1,0}$. Then, the conditions i) and ii) in Theorem A are satisfied. To see the condition iii) in Theorem A more precisely, we choose local fields of unitary frames $e_1, \cdots, e_n$ and e_{n+1} for TM^c and NM^c, respectively. Then,

$$de_i = \sum_{j=1}^{n} \omega_{i\bar{j}} \otimes e_j + [de_i]^N \qquad \text{for} \quad i = 1, \cdots, n ,$$

and

$$de_{n+1} = \omega_{n+1,\overline{n+1}} \otimes e_{n+1} + [de_{n+1}]^T ,$$

where $\omega_{i\bar{j}}$ and $\omega_{n+1,\overline{n+1}}$ are the connection 1-forms for TM^c and NM^c respectively. Therefore, the condition iii) in Theorem A is satisfied if and only if

$$(2.1) \qquad [de_{n+1}] \cdot e_i \equiv 0 \qquad \text{for} \quad i = 1, \cdots, n .$$

We denote by f_A, f_{AB} and f_{ABC}, the first order covariant derivative with respect to e_A, the second order covariant derivative with respect to e_A and e_B, the third order covariant derivative with respect to e_A, e_B and e_C of an immersion f, respectively. From now on, we use the following convention on the range of indices:

$$i, j, k, \cdots = 1, \cdots, n:$$
$$A, B, C, \cdots = 1, \cdots, n, \bar{1}, \cdots, \bar{n} .$$

Then, we can easily see that (2.1) is equivalent to

$$(2.2) \qquad (f_{ij})^{0,1} \equiv 0 , \quad (f_{i\bar{j}})^{0,1} \equiv 0 \qquad \text{for any } i \text{ and } j ,$$

where $(f_{AB})^{0,1}$ is the $NM^{0,1}$-component of f_{AB}. Note that (2.2) is also equivalent to

$$(2.3) \qquad (\partial s)^T \equiv 0 \quad \text{for any} \quad s \in C_0^\infty(NM^{1,0}) .$$

§3. **Proof of (*).**

From Proposition 1, we already know that f satisfies

$$(3.1) \qquad (f_{i\bar{j}})^{0,1} \equiv 0 \quad \text{for any } i \text{ and } j .$$

LEMMA 3.1. *For a fixed vector $a \in C^{2n+2}$, we have*

$$(3.2) \qquad \sum_i D_i D_{\bar{i}} a^{1,0} = -\sum_{i,j} (a \cdot f_{ij})(e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) e_{n+1} ,$$

where D is the normal connection of M and $a^{1,0}$ is the $NM^{1,0}$ component of a.

PROOF. First, note that $f_{AB} \in NM^C$. We may write $a^{1,0} = (a \cdot e_{\overline{n+1}}) e_{n+1}$. Using (3.1), we see that

$$(3.3) \qquad (\partial_{\bar{i}} e_{\overline{n+1}})^T = -\sum_j (e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) f_j ,$$

$$(3.4) \qquad (\partial_i e_{\overline{n+1}})^N = D_i e_{\overline{n+1}} = \omega_{\overline{n+1},n+1}(e_i) e_{\overline{n+1}} .$$

Then, we have

$$\begin{aligned} D_{\bar{i}} a^{1,0} &= (a \cdot (\partial_{\bar{i}} e_{\overline{n+1}})^T) e_{n+1} + (a \cdot D_{\bar{i}} e_{\overline{n+1}}) e_{n+1} \\ &\quad + (a \cdot e_{\overline{n+1}}) D_{\bar{i}} e_{n+1} \\ &= -\sum_j (a \cdot f_j)(e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) e_{n+1} \\ &\quad + \omega_{\overline{n+1},n+1}(e_{\bar{i}})(a \cdot e_{\overline{n+1}}) e_{n+1} + \omega_{n+1,\overline{n+1}}(e_{\bar{i}})(a \cdot e_{\overline{n+1}}) e_{n+1} \\ &= -\sum_j (a \cdot f_j)(e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) e_{n+1} . \end{aligned}$$

Moreover, we have

$$\begin{aligned} (3.5) \qquad \sum_i D_i D_{\bar{i}} a^{1,0} &= -\sum_{i,j} (a \cdot f_{ij})(e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) e_{n+1} \\ &\quad -\sum_{i,j} (a \cdot f_j)\{((\partial_i e_{\overline{n+1}})^N \cdot f_{\bar{i}\bar{j}}) + (e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}i})\} e_{n+1} \\ &\quad -\sum_{i,j} (a \cdot f_j)(e_{\overline{n+1}} \cdot f_{\bar{i}\bar{j}}) D_i e_{n+1} . \end{aligned}$$

Since $(\sum_i f_{\bar{i}\bar{j}i})^N = (\sum_i f_{i\bar{i}\bar{j}})^N = 0$ by Ricci identity and the minimality of f, (3.5), together with (3.4) and the fact that $D_i e_{n+1} = \omega_{n+1,\overline{n+1}}(e_i) e_{n+1}$, yields (3.2). Q.E.D.

We put $s=\lambda\sigma$ in the stability inequality (1.4), where λ is a smooth $\boldsymbol{R}$-valued function with compact support and $\sigma\in C^\infty(NM^C)$. Then, we may rewrite (1.4) as

$$\int_M \lambda^2 \sum_i |(\partial_i\sigma)^T|^2 \leqq \int_M |\lambda_i|^2|\sigma|^2 + \int_M \sum_i \{\lambda\lambda_{\bar{i}}(\sigma\cdot D_i\bar{\sigma}) + \lambda\lambda_i(\bar{\sigma}\cdot D_{\bar{i}}\sigma)\} + \int_M \lambda^2 \sum_i (D_i\bar{\sigma}\cdot D_{\bar{i}}\sigma)\ . \tag{3.6}$$

Since

$$\begin{aligned}0&=\int_M \sum_i \{\partial_{\bar{i}}(\lambda^2(\sigma\cdot D_i\bar{\sigma})/2) + \partial_i(\lambda^2(\bar{\sigma}\cdot D_{\bar{i}}\sigma)/2)\}\\ &=\int_M \sum_i \{\lambda\lambda_{\bar{i}}(\sigma\cdot D_i\bar{\sigma}) + \lambda\lambda_i(\bar{\sigma}\cdot D_{\bar{i}}\sigma)\}\\ &\quad+\int_M \sum_i \lambda^2(D_{\bar{i}}\sigma\cdot D_i\bar{\sigma}) + \int_M \sum_i (\lambda^2/2)((\sigma\cdot D_{\bar{i}}D_i\bar{\sigma}) + (\bar{\sigma}\cdot D_iD_{\bar{i}}\sigma))\ ,\end{aligned}$$

(3.6) may be rewritten as

$$\int_M \sum_i \lambda^2|(\partial_i\sigma)^T|^2 \leqq \int_M \sum_i |\lambda_i|^2|\sigma|^2 - \int_M \sum_i \lambda^2\,\mathrm{Re}\{(\bar{\sigma}\cdot D_iD_{\bar{i}}\sigma)\}\ . \tag{3.7}$$

Note that

$$\sum_i |(\partial_i\sigma)^T|^2 = \sum_{i,j} |\sigma\cdot f_{ij}|^2 \quad\text{and}\quad |d\lambda|^2 = 2\sum_i |\lambda_i|^2\ ,$$

which, together with (3.7), yield

$$2\int_M \lambda^2 \sum_{i,j} |\sigma\cdot f_{ij}|^2 + 2\int_M \lambda^2 \sum_i \mathrm{Re}\{(\bar{\sigma}\cdot D_iD_{\bar{i}}\sigma)\} \leqq \int_M |d\lambda|^2|\sigma|^2\ . \tag{3.8}$$

For a unit vector $a\in\boldsymbol{C}^{2n+2}$, we put $\sigma=a^{1,0}$ in (3.8). Then, by Lemma 3.1, we obtain

$$\int_M \lambda^2 q \leqq \int_M |d\lambda|^2|a^{1,0}|^2 \leqq \int_M |d\lambda|^2\ , \tag{3.9}$$

where

$$q = -2\,\mathrm{Re}\{\sum_{i,j} (a\cdot(f_{ij})^{1,0})(\bar{a}\cdot(f_{\bar{i}\bar{j}})^{1,0})\}\ . \tag{3.10}$$

Since (3.9) holds for all smooth functions λ with compact support, using a theorem of D. Fisher-Colbrie and R. Schoen [2], we see that there exists a smooth function $u>0$ on M such that

$$-\Delta u+qu=0 , \tag{3.11}$$

where

$$\Delta u=-2\sum_i u_{i\bar{i}} .$$

If we put $w=\log u$, then we have

$$\Delta w=q+|dw|^2 . \tag{3.12}$$

Since

$$\begin{aligned} 0&=\int_M \sum_i \{\partial_{\bar{i}}(\lambda^2 w_i)+\partial_i(\lambda^2 w_{\bar{i}})\} \\ &=2\int_M \lambda(d\lambda\cdot dw)-\int_M \lambda^2\Delta w , \end{aligned}$$

from (3.12) we obtain

$$2\int_M \lambda(d\lambda\cdot dw)=\int_M \lambda^2 q+\int_M \lambda^2|dw|^2 . \tag{3.13}$$

Moreover, using $2|\lambda(d\lambda\cdot dw)|\leqq(\lambda^2/2)|dw|^2+2|d\lambda|^2$, we have

$$2\int_M |d\lambda|^2\geqq\int_M \lambda^2 q+\frac{1}{2}\int_M \lambda^2|dw|^2 . \tag{3.14}$$

We choose a unitary basis $a_1, a_2, \cdots, a_{2n+2}$ of C^{2n+2} and we let q_α stand for the expression in (3.10) with $a=a_\alpha$, $\alpha=1, 2, \cdots, 2n+2$. The solution of (3.11) with $q=q_\alpha$ is denoted by u_α and $w_\alpha=\log u_\alpha$. Since we can easily see that

$$\sum_{\alpha=1}^{2n+2} q_\alpha=-2\,\mathrm{Re}\{\sum_{i,j}((f_{ij})^{1,0}\cdot(f_{\bar{i}\bar{j}})^{1,0})\}=0 ,$$

from (3.14) we have

$$\int_M |d\lambda|^2\geqq\int_M \lambda^2 r , \tag{3.15}$$

where

$$r=\frac{1}{8(n+1)}\sum_{\alpha=1}^{2n+2}|dw_\alpha|^2 . \tag{3.16}$$

Again by the theorem of Fischer-Colbrie and Schoen, we see that there exists a smooth function $v>0$ on M such that

$$-\Delta v+rv=0 .$$

Since $r\geqq 0$, we have

$$\Delta v=rv\geqq 0 ,$$

which implies that v is a positive superharmonic function. It follows from the parabolicity of M that $v=$constant. Therefore, we have $r\equiv 0$ and $w_\alpha=$constant, $u_\alpha=$constant, for $\alpha=1, 2, \cdots, 2n+2$. It follows from (3.11) that

$$q_\alpha\equiv 0 \quad \text{for} \quad \alpha=1, 2, \cdots, 2n+2 .$$

For any point $p\in M$, either $(f_{ij})(p)=0$ for any i and j, or $(f_{ij})(p)\neq 0$ for some i and j holds.

Let $L_p=\{(k, l)\,|\,(f_{kl})(p)\neq 0\}$ be the set of the pair of the indices k and l such that $(f_{kl})(p)\neq 0$ at $p\in M$. Then, if we put $a_1=(f_{ij})(p)/|(f_{ij})(p)|$ for any $(i, j)\in L_p$, it follows from $q_1=0$ that

$$\text{(3.17)} \qquad \begin{aligned} &\mathrm{Re}\{\sum_{k,l}((f_{ij})^{0,1}(p)\cdot(f_{kl})^{1,0}(p))((f_{\bar{i}\bar{j}})^{0,1}(p)\cdot(f_{\bar{k}\bar{l}})^{1,0}(p))\} \\ &=0 . \end{aligned}$$

Since (3.17) holds for any $(i, j)\in L_p$, we obtain

$$\mathrm{Re}\{\sum_{i,j,k,l}((f_{ij})^{0,1}\cdot(f_{kl})^{1,0})((f_{\bar{i}\bar{j}})^{0,1}\cdot(f_{\bar{k}\bar{l}})^{1,0})\}=0 ,$$

which implies

$$\text{(3.18)} \qquad \sum_{i,j} f^1_{ij}f^1_{\bar{i}\bar{j}}=0 ,$$

where f^1_{ij} is the component of a vector $(f_{ij})^{1,0}$ with respect to e_{n+1}. We also denote by $f^{\bar{1}}_{ij}$ the component of a vector $(f_{ij})^{0,1}$ with respect to $e_{\overline{n+1}}$. From the Gauss equation we see that the curvature tensor $R=(R_{i\bar{j}k\bar{l}})$ and the scalar curvature τ of M are given respectively by

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -f^1_{ik}f^{\bar{1}}_{\bar{j}\bar{l}}-f^{\bar{1}}_{ik}f^1_{\bar{j}\bar{l}} , \\ \tau &=2\sum_{i,j} R_{i\bar{j}j\bar{i}} \\ &=-2\sum_{i,j}\{|f^1_{ij}|^2+|f^1_{\bar{i}\bar{j}}|^2\} . \end{aligned}$$

Since $|R|^2=4\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}R_{\bar{i}j\bar{k}l}$, the assumption $|R|^2\geqq\tau^2$, together with (3.18) yields

$$(\sum_{i,j}|f^1_{ij}|^2)(\sum_{k,l}|f^{\bar 1}_{\bar k\bar l}|^2)=0 .$$

Therefore, we have proved that for any point $p\in M$, either

$$f^1_{ij}(p)=0 \qquad \text{for any } i \text{ and } j , \tag{3.19}$$

or

$$f^{\bar 1}_{ij}(p)=0 \qquad \text{for any } i \text{ and } j$$

holds.

We need the following

LEMMA 3.2. *$\sum_{i,j}(f_{ij})^{1,0}\otimes(dz^i dz^j)$ (resp. $\sum_{i,j}(f_{ij})^{0,1}\otimes(dz^i dz^j)$) is a holomorphic quadratic differential with the value in $NM^{1,0}$ (resp. $NM^{0,1}$), where (z^i) is a local complex coordinate system on M.*

PROOF. Since NM^C is the holomorphic vector bundle with the connection D over M and D preserves $NM^{1,0}$ and $NM^{0,1}$, it is enough to show that

$$D_{\bar k}(f_{ij})=0 .$$

We see that

$$D_{\bar k}(f_{ij})=(\partial_{\bar k}f_{ij})^N=(f_{ij\bar k})^N=(f_{i\bar k j})^N=0 ,$$

because $f_{i\bar j}=0$ for any i and j. Q.E.D.

Lemma 3.2, together with (3.19), implies that each of f^1_{ij} and $f^{\bar 1}_{ij}$ for any i and j vanishes either identically on M or only at isolated points. Without loss of generality, we may assume that $f^{\bar 1}_{ij}$ vanishes identically for any i and j.

Therefore, Theorem A, (2.2) and (3.1) imply that f is holomorphic with respect to some orthogonal complex structure of R^{2n+2}. This completes the proof of (*).

§4. Proof of Theorem 2.

By Remark in [4], p. 63, Theorem A still holds even if we replace R^{2n+2} by R^{2n+2}/D. Moreover, by the same reason as Theorem 1′ in [4, p. 73], (3.9) is still valid for any $\lambda\in C^\infty(M)$. Hence, we see that the lowest eigenvalue μ_1 of $\Delta-q$ on M is non-negative. Therefore, it follows that if $u>0$ is the eigenfunction of $\Delta-q$ corresponding to μ_1, then we get $\Delta u-qu\geqq 0$. If we put $w=\log u$, then we have $\Delta w\geqq q+|dw|^2$. This implies that (3.14), (3.15) and (3.16) are still valid. If we put $\lambda\equiv 1$ in (3.15), we obtain $r\equiv 0$ and u=constant and therefore $q_\alpha \leqq 0$ for $\alpha=1, 2, \cdots, 2n+2$. This, together with $\sum_\alpha q_\alpha=0$, implies $q_\alpha=0$ for $\alpha=1, 2, \cdots, 2n+2$. Therefore, by the same argument as the proof of (*), we have Theorem 2.

REMARK 5. After the completion of this paper, the author has been informed that M. Dajczer and G. Thorbergsson ("Holomorphicity of minimal submanifolds in complex space forms", preprint) also obtained our Theorem 1 and Proposition 2, independently.

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