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SIMPLENESS AND CLOSEDNESS OF CIRCLES IN COMPACT HERMITIAN SYMMETRIC SPACES

By

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Abstract. We first interpret circles in Riemannian Symmetric space by Lie algebro-theoretic formalism. In particular, it is a solution of the system of ordinary differential equation of first order. We divide circles into 3-types. We investigate closedness and simpleness for such circles in compact Hermitian symmetric spaces. Consequently, we find many open holomorphic circles and non-simple circles. Note that there exist no non-simple circles and no open holomorphic circles in compact Riemannian symmetric space of rank one.

Introduction

Geodesics in Riemannian symmetric spaces $N = G/K$ are well-understood and they are orbits of one parameter subgroups of the full isometry group of N, i.e., are of the form $\exp tX \cdot o$, where $X \in \mathscr{M} \cong T_oN$, $o = \{K\}$. Every geodesic in symmetric space is a simple curve. If N is compact, then N has a simply closed geodesic, and moreover, if N is of rank one, all the geodesics in N are simply closed and have the same prime period (see [H]). The concept of geodesic is extended to higher dimensional case as totally geodesic submanifolds or minimal submanifolds, which are studied systematically by many differential geometers. However, helices have been received less attention. In particular, even circles in symmetric spaces are not studied in detail. Here, we mean by a circle of curvature κ a curve $\gamma(t)$ (parametrized by arc-length t) which satisfies the following equation:

$$\nabla_t\dot{\gamma}(t) = \kappa Y_t \quad \text{and} \quad \nabla_t Y_t = -\kappa\dot{\gamma}(t) \tag{0.1}$$

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for some positive constant κ and a field of unit vector Y_t perpendicular to $\dot{\gamma}(t)$ along γ, where ∇_t is the covariant differentiation along γ with respect to the Riemannian connection ∇ of N (see [NY]). These are precisely the curves with non-zero parallel geodesic curvature vector. When $\kappa = 0$ we can regard the equation (0.1) as the equation for geodesics. From physical point of view, some circles can be interpreted as a motion of a charged particle under an action of a magnetic field (see [C] and also [A2]). We say that a circle $\gamma(t)$ is *closed* if there exists $t_0(\neq 0)$ with $\gamma(t_0) = \gamma(0)$, $\dot{\gamma}(t_0) = \dot{\gamma}(0)$ and $Y_{t_0} = Y_0$. The minimum positive t_0 with such properties is called the *prime period* of the closed circle. When $\gamma(t)$ is an *open* circle, that is, a circle which is not closed, we say it is *simple* if it does not have multiple points, that is, $\gamma(t_1) \neq \gamma(t_2)$ whenever $t_1 \neq t_2$. A closed circle γ is *simple* if $\gamma\,|_{[0,t_0)}$ does not have multiple points, where t_0 denotes the prime period of $\gamma(t)$. In case where N is a complex projective space $\boldsymbol{C}P^n$, the present authors ([AMU]) proved that every circle γ in $\boldsymbol{C}P^n$ (of constant holomorphic sectional curvature 4) is a simple curve and is closed if and only if the complex torsion θ of γ satisfies $\theta = 0$, or $\theta = \pm 1$; or $\theta \neq 0, \pm 1$ and one of the three ratios $a/b, b/c$ and c/a is rational, where a, b, c $(a < b < c)$ are non-zero real solutions of the cubic equation $\lambda^3 - (\kappa^2 + 1)\lambda + \kappa\theta = 0$, where κ is the curvature of γ. The complex torsion θ is defined by $\theta = \langle \dot{\gamma}(t), JY_t \rangle$ for a circle γ in a Kähler manifold $(M, J, \langle\,,\,\rangle)$ and θ is independent of t. In [AM] and [A], the case where N is a complex hyperbolic space or a quaternionic space form is treated. Recently, Mashimo-Tojo ([MT]) proved that any circle $\gamma(t)$ in a Riemannian symmetric space N is an orbit of a 1-parameter subgroup of the full isometry group if and only if N is a symmetric space of rank one or N is a Euclidean space. In these spaces, circles are of the form $\exp t(\kappa H + X) \cdot o$ with $H \in \mathscr{K}$, $X \in \mathscr{M}$, where $\mathscr{K}$ is the Lie algebra of K and $\mathscr{G} = \mathscr{K} + \mathscr{M}$.

In section 1, we reformulate the differential equation of circles in symmetric spaces by using Lie algebraic theory. We take a lift of γ to G and rewrite (0.1) in terms of the Maurer-Cartan form for N. We then derive from the rewritten equation a system of ordinary differential equations of first order. We find it has a solution of power series with infinite radius of convergence due to the Cauchy's method of majorant series. We give two classes of circles, namely those classes of the form $\exp t(\kappa H + X) \cdot o$, $(H \in \mathscr{K}, X \in \mathscr{M})$, which we call *circles of the first kind*, and those of the form $\exp((1/\kappa)(\sin(\kappa t)X + (1 - \cos(\kappa t))Y)) \cdot o$, $(X, Y \in \mathscr{M}$, $[X, Y] = 0)$, which we call *circles of the second kind.*

In section 2, first of all, we define a notion of holomorphic circle (which constitutes a subclass of the class of circle of the first kind when the ambient space is a Hermitian symmetric space) and we prove that every circle of the first kind in a 2-dimensional complex quadric Q^2 is necessarily a holomorphic circle with respect to some invariant complex structure on Q^2. Next, we investigate the closedness of holomorphic circles in complex Grassmann manifolds. We remark that every circle of the first kind is a simple curve. We give an answer to the question "When is a holomorphic circle closed in $G_m(C^{m+n})$?". Consequently, we can find closed circles and open circles of the same curvature, which shows that holomorphic circles in $G_m(C^{m+n})$ cannot be classified up to isometries of the ambient space only by curvature.

Finally, we mention that the congruent classes of circles. In a complex projective space or a complex hyperbolic space two circles with same curvature and same complex torsion are congruent to each other under holomorphic isometries. On the other hand, in a compact Hermitian symmetric space of rank greater than one, there are circles with the same curvatures and the same complex torsions which are not congruent to each other. For example, let γ_1 be a circle of curvature κ which lies on a totally geodesic submanifold $\boldsymbol{R}P^2(c/4)$ in N and γ_2 be a circle of the second kind of the same curvature κ. Note that γ_1 is of the first kind. Although these circles have the same null complex torsion, they are never congruent to each other under the isometry group of N for any κ.

§1. Circles in Riemannian Symmetric Spaces

Let $N = G/K$ be a Riemannian symmetric space with a G-invariant Riemannian metric g. We have the reductive symmetric decomposition of the Lie algebra $\mathscr{G}$ of G as follows:

$$\mathscr{G} = \mathscr{K} + \mathscr{M}, \quad [\mathscr{K}, \mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M}, \mathscr{M}] \subset \mathscr{K},$$

where $\mathscr{K}$ is the Lie algebra of K and $\mathscr{M}$ is identified with the tangent space of N at the base point $o = \{K\}$. We denote by T_xN the tangent space of N at a point $x = g \cdot o \in N$, where $g \in G$ and by $\pi : G \to N$ the projection which is given by $\pi(g) = g \cdot o$. The map $\mathscr{G} \to T_xN$ given by $\xi \to (d/dt)|_{t=0} \exp t\xi \cdot x$ restricts an isomorphism $\operatorname{Ad} g\mathscr{M} \to T_xN$. We denote the inverse map by $\beta_x : T_xN \to \operatorname{Ad} g\mathscr{M} \subset \mathscr{G}$. We may regard β as a $\mathscr{G}$-valued 1-form on N, which is called the *Maurer-Cartan form* of N (see [BR]). We define the bundles $[\mathscr{M}]$ and $[\mathscr{K}]$ with fibres, respectively, $[\mathscr{M}]_x = \operatorname{Ad} g\mathscr{M}$ and $[\mathscr{K}]_x = \operatorname{Ad} g\mathscr{K}$ at $x \in N$, which are subbundles of the trivial bundle $N \times \mathscr{G}$ over N. If $X = (d/dt)|_{t=0} \exp t\xi \cdot x$ then

$$\beta_x(X) = \operatorname{Ad} g P_{\mathscr{M}}(\operatorname{Ad} g^{-1}\xi) \tag{1.1}$$

where $P_{\mathscr{M}} : \mathscr{G} \to \mathscr{M}$ is the projection. Pulling the equation (1.1) back to G, we have

$$(\pi^*\beta)_g = \mathrm{Ad}\, g(P_{\mathscr{M}}\theta) \tag{1.2}$$

where θ is the left-invariant Maurer-Cartan form of G.

Let $\gamma : \boldsymbol{R} \to N$ be a curve. There is a lift $F : \boldsymbol{R} \to G$ of the map γ with $\gamma = \pi \circ F$ (such a lift always exists globally when the domain is simply-connected). By the equation (1.2) we have

$$(\gamma^*\beta)_t = \mathrm{Ad}\, F(t) \cdot \alpha_{\mathscr{M}} \tag{1.3}$$

where $\alpha = F^{-1}\, dF$ and $\alpha_{\mathscr{M}}$ is the $\mathscr{M}$-component of α in the decomposition $\alpha = \alpha_{\mathscr{M}} + \alpha_{\mathscr{K}}$. Let ∇ be the Levi-Civita connection for (N, g). Denoting $P_{[\mathscr{M}]} : N \times \mathscr{G} \to [\mathscr{M}]$ the projection along $[\mathscr{K}]$, we have

$$\beta \circ \nabla = P_{[\mathscr{M}]} \circ d \circ \beta$$

(see [BR], p22). This means that the Levi-Civita connection for the Riemannian symmetric space (N, g) is nothing but the flat differentiation in $N \times \mathscr{G}$ followed by projection. We have the following lemma.

LEMMA 1.1. *A curve $\gamma(t)$ parametrized by arc-length t is a circle of curvature $\kappa(>0)$ if and only if the following equations hold*:

$$\begin{cases} \dfrac{d}{dt}\alpha_{\mathscr{M}}\left(\dfrac{d}{dt}\right) + \left[\alpha_{\mathscr{K}}\left(\dfrac{d}{dt}\right), \alpha_{\mathscr{M}}\left(\dfrac{d}{dt}\right)\right] = L, \\ \dfrac{dL}{dt} + \left[\alpha_{\mathscr{K}}\left(\dfrac{d}{dt}\right), L\right] = -\kappa^2 \alpha_{\mathscr{M}}\left(\dfrac{d}{dt}\right). \end{cases} \tag{1.4}$$

PROOF. It follows from (1.3) that

$$\begin{aligned} \beta \circ \nabla_t \dot{\gamma} &= P_{[\mathscr{M}]} \circ \frac{d}{dt}\beta(\dot{\gamma}) \\ &= \mathrm{Ad}\, F \circ \left\{\frac{d}{dt}\alpha_{\mathscr{M}}\left(\frac{d}{dt}\right) + \left[\alpha_{\mathscr{K}}\left(\frac{d}{dt}\right), \alpha_{\mathscr{M}}\left(\frac{d}{dt}\right)\right]\right\}. \end{aligned}$$

Repeating this process one more time, we have (1.4). q.e.d.

Here, we give some examples which satisfy the equation (1.4).

EXAMPLE 1.1 (Circles of the first kind). When γ is a circle of the first kind, the equation (1.4) is reduced to the following equations:

$$\begin{cases} [H, X] = Y \\ [H, Y] = -X, \end{cases}$$

where $\dot{\gamma}(0) = X, (\nabla_t \dot{\gamma}(t))(0) = \kappa Y$ under the identification $\mathcal{M} \cong T_o N$ by β. When N is a Riemannian symmetric space of rank one, since there is always such $H \in \mathcal{K}$ for any orthonormal two vectors $\{X, Y\}$ with $(\mathrm{ad}\, H)^2 = -1$ on a plane spanned by X and Y, it follows from the uniqueness of the solutions of ordinary differential equation that any circle of curvature κ is of the form $\exp t(\kappa H + X) \cdot o$ (see [MT]). Since, if $\gamma(0) = \gamma(s)$ for some $s > 0$, then $\gamma(t) = \gamma(s + t)$ holds for any t, we see that any circle of the first kind is a simple curve. A Riemannian homogeneous space which admits no circles other than circles of the first kind is necessarily a Riemannian symmetric space of rank one or a Euclidean space(see [MT]).

Example 1.2 (Circles of the second kind). Choose $X, Y \in \mathcal{M}$ with the properties $\|X\|^2 = \|Y\|^2 = 1$, $g(X, Y) = 0$ and $[X, Y] = 0$. We then have $\alpha_{\mathcal{K}}(d/dt) = 0$, $\alpha_{\mathcal{M}}(d/dt) = \cos(\kappa t) \cdot X + \sin(\kappa t) \cdot Y$. Note that $\|\dot{\gamma}(t)\| = \|\mathrm{Ad}\, F(t) \cdot \alpha_{\mathcal{M}}(d/dt)\| = 1$. Set $L = \kappa(-\sin(\kappa t) \cdot X + \cos(\kappa t) \cdot Y)$. We then see that the equation (1.4) holds. In this case, $\dot{\gamma}(0) = \alpha_{\mathcal{M}}(d/dt)|_{t=0} = X$, $(\nabla_t \dot{\gamma}(t))(0) = \kappa Y$ under the identification $\mathcal{M} \cong T_o N$ by β. Henceforth, $\gamma(t)$ is a circle of curvature $\kappa(>0)$ by Lemma 1.1. It is clear that $\gamma(t)$ is a closed curve of prime period $2\pi/\kappa$. Moreover, it is easy to see that $\gamma(t)$ lies on a 2-dimensional totally geodesic flat surface in N. Therefore, if N is compact and admits a 2-dimensional flat torus as a totally geodesic submanifold then $\gamma(t)$ is a non-simple curve if and only if $\kappa \le \sqrt{c}/\pi$, where c is the maximal sectional curvature of N. In particular, when N is a compact Hermitian symmetric space of rank greater than 1 we see that N admits infinitely many non-simple circles.

We now explain the position of the above examples among circles in Riemannian symmetric spaces of compact or non-compact type. In this case, we shall assume that the Riemannian metric g is given by the Killing form B of $\mathcal{G}$. Fix arbitrary $X \in \mathcal{M}$, and set

$$m_1(X) = \{[H, X] \mid H \in \mathcal{K}\}, \quad m_2(X) = \{[H, X] \mid H \in \mathcal{K}, [H, [H, X]] = -X\}.$$

It follows from $g([H, X], Z) = (-1)^s B(H, [X, Z])$, where $s = 0$ or 1 according as N is of noncompact or compact type, that the orthogonal complement $m_1^{\perp}(X)$ of $m_1(X)$ in $\mathcal{M}$ with respect to g is given by

$$m_1^{\perp}(X) = \{Z \in \mathcal{M} \mid [X, Z] = 0\}.$$

Therefore, if $Y \in m_2(X)$ then the circle is of the first kind, and if $Y \in m_1^{\perp}(X)$ then the circle is of the second kind. Note that $m_1^{\perp}(X)$ is equal to the curvature nullity space $N(X) = \{Z \in \mathscr{M} \mid g(R(X,Z)Z,X) = 0\}$, where R is the curvature tensor of (N,g). Thus, the remaining class of circles in a Riemannian symmetric space consists of circles which have the property $\dot{\gamma}(0) = X$, $(\nabla_t\dot{\gamma}(t))(0) = \kappa Y$ and $Y = aZ + bW$ with $Z \in m_1^{\perp}(X)$, $W \in m_1(X)$, $a^2 + b^2 = 1$, where $0 < |b| < 1$, or $|b| = 1$ and $W \in m_1(X)\backslash m_2(X)$. We call the circle of this kind *a circle of the general kind.*

An equation of the circles of curvature κ in N is now interpreted by the following system of ordinary differential equations of first order:

THEOREM 1.1. *Represent F as $F = gh$, where $h : \boldsymbol{R} \to K$ and g has a property that $g^{-1}(dg/dt) \in \mathscr{M}$. Then, $\gamma = \pi \circ F$ is a circle of curvature κ with initial conditions $\dot{\gamma}(0) = X$, $(\nabla_t\dot{\gamma}(t))(0) = \kappa Y$ if and only if g is a solution of the following differential equation*

$$g^{-1}\frac{dg}{dt} = \cos(\kappa t)X + \sin(\kappa t)Y \quad \text{with } g(0) = I. \tag{1.5}$$

Moreover, each entry of the solution $g(t)$ is represented by the power series of t with infinite radius of convergence.

PROOF. For $F = gh$, we have

$$F^{-1}\frac{dF}{dt} = \operatorname{Ad} h^{-1}\left(g^{-1}\frac{dg}{dt}\right) + h^{-1}\frac{dh}{dt}.$$

Therefore, by our assumption on the choice of g we have

$$\alpha_{\mathscr{M}} = \operatorname{Ad} h^{-1}\left(g^{-1}\frac{dg}{dt}\right), \quad \alpha_{\mathscr{K}} = h^{-1}\frac{dh}{dt}.$$

Set $\tilde{\alpha}_{\mathscr{M}} = \operatorname{Ad} h\alpha_{\mathscr{M}}$. Then, differentiating this equation, we obtain

$$\frac{d\tilde{\alpha}_{\mathscr{M}}}{dt} = \operatorname{Ad} h\left\{\frac{d\alpha_{\mathscr{M}}}{dt} + [\alpha_{\mathscr{K}}, \alpha_{\mathscr{M}}]\right\}.$$

Again by differentiating this equation, from Lemma 1.1, we find that $\gamma = \pi \circ F$ is a circle of curvature κ if and only if $d^2\tilde{\alpha}_{\mathscr{M}}/dt^2 = -\kappa^2\tilde{\alpha}_{\mathscr{M}}$. This and the initial conditions imply that

$$\tilde{\alpha}_{\mathscr{M}} = \cos(\kappa t)X + \sin(\kappa t)Y.$$

Since $\tilde{\alpha}_{\mathscr{M}} = g^{-1}(dg/dt)$, we have (1.5). Now, the general theory of ordinary differential equation implies that the solution of (1.5) always exists for all t (if necessary, by taking the real and imaginary part of each entry we may transfer (1.5) to the real-valued equation of the same type). Moreover, since all the coefficients in the right hand side of (1.5) are represented by power series of t with infinite radius of convergence, it follows from the Cauchy's method of majorant series that the solution is also represented by power series of t with infinite radius of convergence. q.e.d.

REMARK. (1) For the circle of the first kind, we take $F = \exp t(\kappa H + X)$, $g = \exp t(\kappa H + X) \cdot \exp t(-\kappa H)$ and $h = \exp t(\kappa H)$. For the circle of the second kind, we take $F = g = \exp((1/\kappa)\sin(\kappa t)X + (1/\kappa)(1 - \cos(\kappa t))Y)$ and $h = I$.

(2) Since a differential equation $h^{-1}(dh/dt) = \alpha_{\mathscr{K}}$ always has a solution h, we see that $g = Fh^{-1}$ always satisfies $g^{-1}(dg/dt) \in \mathscr{M}$ for such a choice of h (cf. p69 of [KN]).

§2. When Is a Holomorphic Circle Closed in $G_m(C^{m+n})$?

First of all, we review some fundamental facts on the geometry of complex Grassmann manifold. Let $G_m(C^{m+n})$ be a complex Grassmann manifold of m-dimensional complex subspaces in C^{m+n}. In this case, $G = SU(m+n)$, $K = S(U(m) \times U(n))$. Moreover, we have

$$\mathscr{M} = \left\{ \begin{pmatrix} \mathbf{0} & -A^* \\ A & \mathbf{0} \end{pmatrix} \middle| A \text{ is an } n \times m - \text{complex matrix} \right\},$$

where A^* is the adjoint matrix of A and $\mathbf{0}$ is an appropriate square zero-matrix. Let $HM(m+n)$ be the space of all Hermitian matrices of order $m+n$. Define $E_1 \in HM(m+n)$ by

$$E_1 = \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Here I_m is the identity matrix of order m. Define a map $\tilde{\rho} : SU(m+n) \to HM(m+n)$ by $\tilde{\rho}(g) = gE_1g^{-1}$. The map $\tilde{\rho}$ induces an injective map $\rho : SU(m+n)/S(U(m) \times U(n)) \to HM(m+n)$. Since ρ is an immersion at the base point o and ρ is G-equivariant, we see that ρ is an embedding. We give $HM(m+n)$ a Hermitian trace metric h with respect to which ρ is an isometric embedding. If we define h by $h(A,B) = (2/c)\,\mathrm{tr}(AB^*)$ for $A, B \in HM(m+n)$ then $G_m(C^{m+n})$ with the induced metric ρ^*h has a maximal sectional curvature $c > 0$.

Let $H_0 = \begin{pmatrix} -(n\sqrt{-1}/(m+n))I_m & \mathbf{0} \\ \mathbf{0} & (m\sqrt{-1}/(m+n))I_n \end{pmatrix}$ be an element of the center of $\mathscr{K}$. Then, an invariant complex structure J of $G_m(\mathbf{C}^{m+n})$ is given by $J_o = \pm \operatorname{ad} H_0$ at o.

In case of circles in compact Riemannian symmetric space of rank one, it is possible to determine the closedness of the circle by solving the cubic eigen-equation. The reason why the closedness of the circle is determined by such a simple equation is that any circle is contained in a totally geodesic complex submanifold $\mathbf{C}P^2$ (see [AMU]). On the other hand, it is not so easy to determine the closedness of circles of the first kind even in $G_2(\mathbf{C}^4)$. In fact, there are circles of the first kind fully embedded in $G_2(\mathbf{C}^4)$. For example, take

$$X = \begin{pmatrix} \mathbf{0} & -A^* \\ A & \mathbf{0} \end{pmatrix}, \quad H = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix}$$

and

$$A = \frac{\sqrt{c}}{\sqrt{28}} \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 2\sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

We then have $[H,[H,X]] = -X$. Set

$$Y = [H,X], \quad Z = [[X,Y],X], \quad W = [[X,Y],Y],$$

$$L = [[X,Y],Z], \quad S = [[X,Z],X], \quad T = [[X,W],X], \quad U = [[X,Y],W].$$

Then, $\operatorname{Span}_{\mathbf{R}}\{X,Y,Z,W,L,S,T,U\}$ is an 8-dimensional subspace of $\mathscr{M}$. Henceforth, a circle $\gamma(t) = \exp t(\kappa H + X)\cdot o$ is fully embedded in $G_2(\mathbf{C}^4)$.

Therefore, we investigate the closedness of circles in the special class, i.e., holomorphic circles.

DEFINITION. A curve $\gamma(t)$ parametrized by arc-length t in a Kähler manifold M is said to be a *holomorphic circle* of curvature κ if $\nabla_t\dot{\gamma}(t) = \pm\kappa J\dot{\gamma}(t)$, where J is the complex structure tensor of M.

In case where M is an irreducible Hermitian symmetric space, an invariant complex structure on M is given at o by $J_o = \pm ad(H_0)$ for some element of the center of $\mathscr{K}$. Thus, in this case, any holomorphic circle belongs to the class of circles of the first kind.

In case where $M = \mathbf{Q}^2 = \mathbf{C}P^1(c) \times \mathbf{C}P^1(c)$, there are two choices of H_0 as follows:

$$H_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We have

THEOREM 2.1. *Any circle $\gamma(t)$ of the first kind in $\boldsymbol{Q}^2$ is a holomorphic circle with respect to $J_o = \pm \operatorname{ad} H_0$ for some choice of $H_0 \in \mathscr{K}$. Moreover, for a circle of curvature κ with $\dot{\gamma}(0) = X = \begin{pmatrix} \mathbf{0} & -{}^tA \\ A & \mathbf{0} \end{pmatrix} \in \mathscr{M}$, where A is a real square matrix of order 2, we have the following*:

(1) *If* $\operatorname{rank} A < 2$ *then $\gamma(t)$ lies on $\boldsymbol{CP}^1(c)$ and it is a simple closed curve of prime period $2\pi/\sqrt{\kappa^2 + c}$.*

(2) *If ${}^tAA = (c/8)I_2$ then $\gamma(t)$ lies on $\boldsymbol{Q}^1 = \boldsymbol{CP}^1(c/2)$ and it is a simple closed curve of prime period $2\pi/\sqrt{\kappa^2 + c/2}$.*

(3) *Otherwise, let α_1, α_2 be the eigenvalues of ${}^tAA$. Then $\gamma(t)$ is a simple closed curve if and only if the ratio $\sqrt{\kappa^2 + 4\alpha_1}/\sqrt{\kappa^2 + 4\alpha_2}$ is a rational number. In this case, the prime period is the least common integral multiple of $2\pi/\sqrt{\kappa^2 + 4\alpha_1}$ and $2\pi/\sqrt{\kappa^2 + 4\alpha_2}$.*

PROOF. Any element $H \in \mathscr{K}$ is represented by

$$H = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad (a, b \in \boldsymbol{R}).$$

For any $X = \begin{pmatrix} \mathbf{0} & -{}^tA \\ A & \mathbf{0} \end{pmatrix}$, where $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x, y, z, w \in \boldsymbol{R}$ and $x^2 + y^2 + z^2 + w^2 = c/4$, if $\gamma(t) = \exp t(\kappa H + X) \cdot o$ is a circle of curvature κ then we must have $[H, [H, X]] = -X$, which is equivalent to the following equations:

$$\text{(2.1)} \qquad \begin{cases} (a^2 + b^2)x - 2abw = x, & (a^2 + b^2)z + 2aby = z, \\ (a^2 + b^2)y + 2abz = y, & (a^2 + b^2)w - 2abx = w. \end{cases}$$

The normalization $\|X\|^2 = \|[H, X]\|^2$ means that $c/4 = (c/4)(a^2+b^2) - 4ab \det A$. Therefore, if $\operatorname{rank} A < 2$ then we have $a^2 + b^2 = 1$, which, together with (2.1), yields $ab = 0$. Hence $\operatorname{ad} H$ is an invariant complex structure of $\boldsymbol{Q}^2$. Next suppose that $\operatorname{rank} A = 2$. It follows from (2.1) that if $y^2 \neq z^2$ or $x^2 \neq w^2$ then $a^2 + b^2 = 1$ and $ab = 0$, hence $\operatorname{ad} H$ is an invariant complex structure. Thus, the remaining case is given by

$$\begin{cases} y = z \\ x = -w \\ (a+b)^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} y = -z \\ x = w \\ (a-b)^2 = 1 \end{cases}$$

because the case where $y = z$, $x = w$ or $y = -z$, $x = -w$ implies that $a^2 + b^2 = 1$ and $ab = 0$. However, a circle for the remaining case is also a holomorphic circle because we have $[H, X] = \pm J_o X$. Moreover, since ${}^tAA = (c/8)I_2$ the corresponding circle lies on Q^1. This fact and the rest of our claim follows from more general result Theorem 2.2 below. q.e.d.

Since any compact Hermitian symmetric space of rank greater than one admits Q^2 as a totally geodesic Kähler submanifold, we obtain the following:

COROLLARY 2.1. *On a compact Hermitian symmetric space of rank $r(\geq 2)$, there exist closed holomorphic circles and open holomorphic circles of any given curvature.*

In the following, we determine the closedness or non-closedness of holomorphic circles in $G_m(C^{m+n})$. We denote by $CP^n(r)$ an n-dimensional complex projective space of constant holomorphic sectional curvature $r(>0)$.

THEOREM 2.2. *Let $\gamma(t) = \exp t(\kappa H_0 + X) \cdot o$ be a holomorphic circle of curvature $\kappa(>0)$ in $G_m(C^{m+n})$ with maximal sectional curvature c.*

(1) *If A^*A is unitary equivalent to $c/(4l)\begin{pmatrix} I_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ for some l $(1 \leq l \leq m)$, then $\gamma(t)$ lies on $CP^1(c/l)$ and is a simple closed curve of prime period $2\pi/\sqrt{\kappa^2 + c/l}$.*

(2) *In general, let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be the non-zero eigenvalues of A^*A which are different from each other. Then $\gamma(t)$ is a simple closed curve if and only if each ratio $\sqrt{\kappa^2 + 4\alpha_j}/\sqrt{\kappa^2 + 4\alpha_k}$ for $j, k = 1, 2, \ldots, s$ is a rational number. In this case, the prime period is the least common integral multiple of $2\pi/\sqrt{\kappa^2 + 4\alpha_1}, \ldots, 2\pi/\sqrt{\kappa^2 + 4\alpha_s}$.*

PROOF. First of all, without loss of generality, we may suppose that

$$H_0 = \begin{pmatrix} -(n/(m+n))\sqrt{-1}I_m & \mathbf{0} \\ \mathbf{0} & (m/(m+n))\sqrt{-1}I_n \end{pmatrix}.$$

The initial tangent vector is $X = \begin{pmatrix} \mathbf{0} & -A^* \\ A & \mathbf{0} \end{pmatrix}$, where A is an $n \times m$-complex matrix with $\mathrm{tr}(A^*A) = c/4$. Since $J_oX = \begin{pmatrix} \mathbf{0} & iA^* \\ iA & \mathbf{0} \end{pmatrix}$, we have $[[X, J_oX], X] = 4\begin{pmatrix} \mathbf{0} & iA^*AA^* \\ iAA^*A & \mathbf{0} \end{pmatrix}$. If A^*A is unitary equivalent to $c/(4l)\begin{pmatrix} I_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, then we see that $A^*AA^* = c/(4l)A^*$ and $g(R(X, J_oX)J_oX, X) = g([[X, J_oX], X], J_oX) = c/l$. Note that the prime period of circle of curvature κ in $CP^1(c/l)$ is equal to $2\pi/\sqrt{\kappa^2 + c/l}$. Next we show (2). Set $V = \kappa H_0 + X$. Let $P = \begin{pmatrix} \boldsymbol{u}_1^f & \boldsymbol{u}_2^f & \cdots & \boldsymbol{u}_{m+n}^f \\ \boldsymbol{u}_1^b & \boldsymbol{u}_2^b & \cdots & \boldsymbol{u}_{m+n}^b \end{pmatrix} \in U(m+n)$ be a unitary matrix such that each $\begin{pmatrix} \boldsymbol{u}_j^f \\ \boldsymbol{u}_j^b \end{pmatrix}$ is an eigenvector of V with eigenvalue λ_j for $j = 1, 2, \ldots, m+n$, where $\boldsymbol{u}_j^f$ (resp. $\boldsymbol{u}_j^b$) is a complex m-dimensional (resp. n-dimensional)column vector. Then the condition $V\begin{pmatrix} \boldsymbol{u}_j^f \\ \boldsymbol{u}_j^b \end{pmatrix} = \lambda_j \begin{pmatrix} \boldsymbol{u}_j^f \\ \boldsymbol{u}_j^b \end{pmatrix}$ leads to

$$(2.2) \qquad \begin{cases} -\dfrac{n\sqrt{-1}}{m+n}\kappa \boldsymbol{u}_j^f - A^*\boldsymbol{u}_j^b = \lambda_j \boldsymbol{u}_j^f \\[2ex] A\boldsymbol{u}_j^f + \dfrac{m\sqrt{-1}}{m+n}\kappa \boldsymbol{u}_j^b = \lambda_j \boldsymbol{u}_j^b \quad (j = 1, 2, \ldots, m+n). \end{cases}$$

Therefore, we obtain $(A^*A)\boldsymbol{u}_j^f = -(\lambda_j - m\sqrt{-1}\kappa/(m+n))(\lambda_j + n\sqrt{-1}\kappa/(m+n))\boldsymbol{u}_j^f$. Hence, the eigenvalues of V consist of the set

$$\left\{ \frac{m\sqrt{-1}\kappa}{m+n}, -\frac{n\sqrt{-1}\kappa}{m+n}, \frac{\sqrt{-1}}{2}\left(\frac{m-n}{m+n}\kappa \pm \sqrt{\kappa^2 + 4\alpha_1}\right), \ldots, \right.$$
$$\left. \frac{\sqrt{-1}}{2}\left(\frac{m-n}{m+n}\kappa \pm \sqrt{\kappa^2 + 4\alpha_s}\right) \right\}.$$

Denoting by $\langle\, , \rangle$ the usual Hermitian inner product on C^m or C^n and noting that $\langle \boldsymbol{u}_j^b, \boldsymbol{u}_l^b \rangle = \delta_{jl} - \langle \boldsymbol{u}_j^f, \boldsymbol{u}_l^f \rangle$ for $1 \leq j,\ l \leq m+n$, we obtain by (2.2) that

$$\begin{cases} \lambda_j \langle \boldsymbol{u}_j^f, \boldsymbol{u}_l^f \rangle = \left(\dfrac{m-n}{m+n}\sqrt{-1}\kappa + \overline{\lambda_l}\right)\langle \boldsymbol{u}_j^f, \boldsymbol{u}_l^f \rangle \quad \text{for } j \neq l, \\[2ex] \left(\lambda_j + \dfrac{n}{m+n}\sqrt{-1}\kappa\right)\|\boldsymbol{u}_j^f\|^2 = -\left(\overline{\lambda_j} + \dfrac{m}{m+n}\sqrt{-1}\kappa\right)\|\boldsymbol{u}_j^b\|^2 \quad \text{for } j = 1, 2, \ldots, m+n. \end{cases}$$

These imply that $\lambda_j \neq (m/(m+n))\sqrt{-1}\kappa$ if and only if $\boldsymbol{u}_j^f \neq 0$. Moreover, if $\langle \boldsymbol{u}_j^f, \boldsymbol{u}_k^f \rangle \neq 0$ for $j \neq k$ then $\lambda_j + \lambda_k = ((m-n)/(m+n))\sqrt{-1}\kappa$ holds. Therefore, if $\langle u_j^f, u_k^f \rangle \neq 0$ for $j \neq k$ then we must have $\{\lambda_j, \lambda_k\} = \{(\sqrt{-1}/2) \cdot (((m-n)/(m+n))\kappa + \sqrt{\kappa^2 + 4\alpha_d}),\ (\sqrt{-1}/2)(((m-n)/(m+n))\kappa - \sqrt{\kappa^2 + 4\alpha_d})\}$ for some $1 \leq d \leq s$. On the other hand, the eigenvectors which correspond to the eigenvalues $\{-(n/(m+n))\sqrt{-1}\kappa, (\sqrt{-1}/2)(((m-n)/(m+n))\kappa + \sqrt{\kappa^2 + 4\alpha_1}), \dots, (\sqrt{-1}/2)(((m-n)/(m+n))\kappa + \sqrt{\kappa^2 + 4\alpha_s})\}$ counting with their multiplicities form an orthonormal basis for $\boldsymbol{C}^m$. But, since the eigenvector corresponding to the eigenvalue $(\sqrt{-1}/2)(((m-n)/(m+n))\kappa - \sqrt{\kappa^2 + 4\alpha_d})$ is orthogonal to any vector of the basis except the eigenvector corresponding to the eigenvalue $(\sqrt{-1}/2)(((m-n)/(m+n))\kappa + \sqrt{\kappa^2 + 4\alpha_d})$, for any d with $1 \leq d \leq s$ we see that a pair of the eigenvalues with the inner product of the corresponding eigenvectors being non-zero must be of the form $\{(\sqrt{-1}/2)(((m-n)/(m+n))\kappa + \sqrt{\kappa^2 + 4\alpha_d}), (\sqrt{-1}/2)(((m-n)/(m+n))\kappa - \sqrt{\kappa^2 + 4\alpha_d})\}$. Now, $\gamma(0) = \gamma(t_0)$ is equivalent to $\mathrm{Ad}\exp t_0 V \cdot E_1 = E_1$. Set $Z_1 = P^{-1} E_1 P$ and $D_t = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_{m+n} t})$, where $diag(a_1, a_2, \dots, a_{m+n})$ is a diagonal matrix with $a_1, a_2, \dots, a_{m+n}$ as the diagonal elements. Then, we see that $\gamma(0) = \gamma(t_0)$ is equivalent to $D_{t_0} Z_1 = Z_1 D_{t_0}$, which is also equivalent to $(\lambda_j - \lambda_k)t_0 \in 2\pi i \boldsymbol{Z}$ for $\langle u_j^f, u_k^f \rangle \neq 0$. This means that $\sqrt{\kappa^2 + 4\alpha_d}\, t_0 \in 2\pi \boldsymbol{Z}$ for $1 \leq d \leq s$, so that $\sqrt{\kappa^2 + 4\alpha_j}/\sqrt{\kappa^2 + 4\alpha_k}$ is rational for $1 \leq j, k \leq s$. In this case, the prime period is equal to the least common integral multiple of $2\pi/\sqrt{\kappa^2 + 4\alpha_1}, \dots, 2\pi/\sqrt{\kappa^2 + 4\alpha_s}$. q.e.d.

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