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# Geometry of ordinary helices in a complex projective space

(Dedicated to Professor Katsuei Kenmotsu on the occasion of his 60th birthday)

Toshiaki ADACHI, Sadahiro MAEDA and Seiichi UDAGAWA

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**Abstract.** It is well-known that every ordinary helix in Euclidean space is an open curve without self-intersection points. In this paper we study ordinary helices on Riemannian homogeneous spaces. We present an example of closed ordinary helices in a complex projective plane with 6 self-intersection points. We also characterize real space forms in terms of ordinary helices.

*Key words*: ordinary helix, complex projective plane, self-intersection points.

## 1. Introduction

Investigating *nice* curves is important in Differential Geometry as well as in Topology. We pay attention to ordinary helices in a Euclidean 3-space $\mathbb{R}^3$. It is well-known that each ordinary helix $\gamma = \gamma(s)$ parametrized by its arclength $s$ is an open curve without self-intersection points and satisfies the following ordinary differential equations with respect to $s$, which are called the Frenet formula for $\gamma$:

$$\ddot{\gamma} = \kappa V_2, \quad \dot{V}_2 = -\kappa V_1 + \tau V_3, \quad \dot{V}_3 = -\tau V_2. \tag{1.1}$$

Here $\kappa$ and $\tau$ are positive constants, and $\{V_1 = \dot{\gamma}, V_2, V_3\}$ is an orthonormal frame along $\gamma$. We call those constants $\kappa$ and $\tau$ the curvature and the torsion of $\gamma$, respectively, and $\{V_1, V_2, V_3\}$ its Frenet frame. The unit vectors $V_2$ and $V_3$ are called the unit principal normal vector and the unit binormal vector on the curve $\gamma$, respectively.

On a Riemannian manifold $(M, \langle\ ,\ \rangle)$ with Riemannian metric $\langle\ ,\ \rangle$, using the covariant differentiation $\nabla$ induced from the metric $\langle\ ,\ \rangle$, we can consider an ordinary helix $\gamma$ as a solution of a system of ordinary differential

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equations corresponding to (1.1). Needless to say, on every Riemannian manifold $M$, for an arbitrary pair $(\kappa, \tau)$ of two positive constants there exists an ordinary helix on $M$ with curvature $\kappa$ and torsion $\tau$. However, in general such ordinary helices are not necessarily congruent each other under the action of the isometry group $G$ of $M$.

In this paper we study ordinary helices in a nonflat $n$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c$, that is, $M_n(c)$ is complex analytically isometric to either a complex projective space $\mathbb{C}P^n(c)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ endowed with the Bergman metric of constant holomorphic sectional curvature $c$. After reviewing the definitions and some results on ordinary helices we first construct an example of closed ordinary helices in a complex projective plane $\mathbb{C}P^2(c)$ with 6 self-intersection points (Theorem 1). Later in the class of Riemannian homogeneous spaces we provide a characterization of an $n$-dimensional real space form $M^n(c)$ of constant curvature $c$ in terms of ordinary helices (Theorem 2). Here $M^n(c)$ is isometric to a Euclidean space $\mathbb{R}^n$, a standard sphere $S^n(c)$ with radius $1/\sqrt{c}$ or a real hyperbolic space $H^n(c)$ of constant sectional curvature $c$.

## 2. Ordinary helices in a Riemannian manifold

Let $M$ be an $n$-dimensional Riemannian manifold with Riemannian connection $\nabla$. We shall start by reviewing the definitions of circles and ordinary helices in $M$. A smooth curve $\gamma = \gamma(s)$ on $M$ parametrized by its arclength $s$ is called a *circle*, if there exists a field of unit vectors $V_2(s)$ along $\gamma$ which satisfies, together with the field of unit tangent vectors $V_1 = \dot{\gamma}$, the differential equations:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa V_2, \quad \nabla_{\dot{\gamma}} V_2 = -\kappa V_1,$$

where $\kappa$ is a positive constant, which is called the curvature of $\gamma$, and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along $\gamma$ with respect to $\nabla$. We call a smooth curve $\gamma = \gamma(s)$ on $M^n$ with $n \geqq 3$ an *ordinary helix* if there exist an orthonormal frame $\{V_1 = \dot{\gamma}, V_2, V_3\}$ along $\gamma$ and positive constants $\kappa$, $\tau$ which satisfy the following system of ordinary equations

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma &= \kappa V_2, \\ \nabla_{\dot\gamma}V_2 &= -\kappa V_1 + \tau V_3, \\ \nabla_{\dot\gamma}V_3 &= -\tau V_2. \end{cases} \tag{2.1}$$

The orthonormal frame $\{V_1, V_2, V_3\}$ is called the Frenet frame of $\gamma$, those constants $\kappa$ and $\tau$ are called the curvature and the torsion of $\gamma$, respectively. Note that if $M$ is a real analytic Riemannian manifold then every ordinary helix is a real analytic curve. The following is a well-known result on the existence and uniqueness of ordinary helices on a Riemannian manifold.

**Proposition A** *Given a point $x \in M$, an orthonormal system $\{v_1, v_2, v_3\}$ of the tangent space $T_xM$ and two positive constants $\kappa$, $\tau$, we have a unique ordinary helix $\gamma = \gamma(s)$ on $M$ having curvature $\kappa$ and torsion $\tau$ with $\gamma(0) = x$ whose initial frame $(V_1(0), V_2(0), V_3(0))$ is $(v_1, v_2, v_3)$.*

This guarantees the following result on ordinary helices on a real space form, which is either a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$, or a real hyperbolic space $H^n(c)$.

**Proposition 1** *For an arbitrary pair $(\kappa, \tau)$ of positive constants there exists a unique ordinary helix on a real space form $M^n(c)$ with curvature $\kappa$ and torsion $\tau$ up to the action of the isometry group of $M^n(c)$.*

We shall call two ordinary helices $\gamma_1$, $\gamma_2$ on $M$ are congruent under the action of the isometry group of $M$ if there exists an isometry $\varphi$ with $\gamma_2 = \varphi \circ \gamma_1$. This proposition shows that on a real space form $M^n(c)$ we have a one-to-one correspondence between the set $\{(\kappa, \tau) \mid \kappa, \tau > 0\}$ in $\mathbb{R}^2$ and the set of all congruency classes of ordinary helices under the action of the isometry group.

Here we study ordinary helices on a real space form in connection with the action of the isometry group. On a Euclidean space $\mathbb{R}^3$ the equation (2.1) is equivalent to the equation $\gamma^{(4)} + (\kappa^2 + \tau^2)\ddot\gamma = 0$. If we define an isometry $\varphi_s$ of $\mathbb{R}^3$ by $\varphi_s(x) = A(s) \cdot x + v(s)$, where

$$A(s) = \begin{pmatrix} \cos\sqrt{\kappa^2+\tau^2}s & -\sin\sqrt{\kappa^2+\tau^2}s & 0 \\ \sin\sqrt{\kappa^2+\tau^2}s & \cos\sqrt{\kappa^2+\tau^2}s & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3),$$

$$v(s) = \begin{pmatrix} 0 \\ 0 \\ \tau s/\sqrt{\kappa^2+\tau^2} \end{pmatrix},$$

every ordinary helix $\gamma$ is congruent to an ordinary helix $s \mapsto \varphi_s\big({}^t(\kappa/(\kappa^2+\tau^2), 0, 0)\big)$.

Next we represent a standard sphere $S^3(1)$ of curvature 1 as a unit sphere in $\mathbb{R}^4$. Regarding an ordinary helix $\gamma$ on $S^3(1)$ as a curve on $\mathbb{R}^4$, we find the equation (2.1) is equivalent to the equation $\gamma^{(4)} + (\kappa^2+\tau^2+1)\ddot{\gamma} + \tau^2\gamma = 0$. Hence if we define an isometry $\varphi_s$ of $S^3(1)$ by $\varphi_s(x) = A(s)\cdot x$ with

$$A(s) = \begin{pmatrix} \cos\alpha s & -\sin\alpha s & 0 & 0 \\ \sin\alpha s & \cos\alpha s & 0 & 0 \\ 0 & 0 & \cos\beta s & -\sin\beta s \\ 0 & 0 & \sin\beta s & \cos\beta s \end{pmatrix} \in \mathrm{SO}(4),$$

where

$$\alpha = \frac{1}{2}\big(\sqrt{\kappa^2+(\tau+1)^2} + \sqrt{\kappa^2+(\tau-1)^2}\big),$$

$$\beta = \frac{1}{2}\big(\sqrt{\kappa^2+(\tau+1)^2} - \sqrt{\kappa^2+(\tau-1)^2}\big),$$

we see $\gamma$ is congruent to an ordinary helix $s \mapsto \varphi_s\big({}^t(a, 0, b, 0)\big)$ with

$$a = \frac{\sqrt{(\beta^2-1)^2+\kappa^2}}{\alpha^2-\beta^2}, \quad b = \frac{\sqrt{(\alpha^2-1)^2+\kappa^2}}{\alpha^2-\beta^2}.$$

Finally we represent a real hyperbolic space $H^3(-1)$ of curvature $-1$ as a subset $\{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = -1\}$ of a de Sitter space $\mathbb{R}^4_1$ equipped with the inner product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ for $x = {}^t(x_0, x_1, x_2, x_3)$ and $y = {}^t(y_0, y_1, y_2, y_3)$. Regarding an ordinary helix $\gamma$ on $H^3(-1)$ as a curve on $\mathbb{R}^4_1$, we find the equation (2.1) is equivalent to the equation $\gamma^{(4)} + (\kappa^2+\tau^2-1)\ddot{\gamma} - \tau^2\gamma = 0$. Hence if we define an isometry $\varphi_s$ of $H^3(-1)$ by $\varphi_s(x) = A(s)\cdot x$ with

$$A(s) = \begin{pmatrix} \cosh\alpha s & \sinh\alpha s & 0 & 0 \\ \sinh\alpha s & \cosh\alpha s & 0 & 0 \\ 0 & 0 & \cos\beta s & -\sin\beta s \\ 0 & 0 & \sin\beta s & \cos\beta s \end{pmatrix} \in \mathrm{O}(1,3),$$

where

$$\alpha = \frac{\sqrt{2}}{2}\sqrt{\sqrt{(\kappa^2+\tau^2+1)^2+4\tau^2}+1-\kappa^2-\tau^2},$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\kappa^2+\tau^2-1+\sqrt{(\kappa^2+\tau^2+1)^2+4\tau^2}},$$

we see $\gamma$ is congruent to an ordinary helix $s \mapsto \varphi_s({}^t(a,0,b,0))$ with

$$a = \frac{\sqrt{(\beta^2+1)^2-\kappa^2}}{\alpha^2+\beta^2}, \quad b = \frac{\sqrt{\kappa^2-(\alpha^2-1)^2}}{\alpha^2+\beta^2}.$$

**Proposition 2** *Every ordinary helix $\gamma = \gamma(s)$ on a real space form $M^n(c)$ is an integral curve of some Killing vector field on $M^n(c)$, that is, there exists a one-parameter subgroup $\{\varphi_s\}$ of the isometry group of $M^n(c)$ with $\gamma(s) = \varphi_s(\gamma(0))$ for all $s$.*

We now concern ourselves on ordinary helices on a complex space form. We remark that corresponding assertions in Propositions 1 and 2 do not hold for ordinary helices on a nonflat complex space form $M_n(c)$ with complex structure $J$. In order to explain this fact in detail we use the complex torsions $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ of an ordinary helix $\gamma$, where $1 \leqq i < j \leqq 3$ and $\{V_1, V_2, V_3\}$ is the Frenet frame for $\gamma$. Note that every complex torsion $\tau_{ij} = \tau_{ij}(s)$ of each ordinary helix is a real analytic function and satisfies $|\tau_{ij}(s)| \leqq 1$ for every $s$. In this paper we shall call a smooth curve $\gamma$ on $M_n(c)$, $c \neq 0$ a *Killing ordinary helix* if it is an ordinary helix and also is an integral curve of some Killing vector field on $M_n(c)$. For the later use we recall the following.

**Theorem A** ([MA]) *Let $M_n(c)$ be an $n\,(\geqq 3)$-dimensional nonflat complex space form. Then the following hold:*

(1) *All complex torsions of each Killing ordinary helix $\gamma$ in $M_n(c)$ are constant. They satisfy*

$$\kappa\tau_{23} = \tau\tau_{12}, \quad \tau_{13} = 0, \quad |\tau_{12}| \leqq \kappa/\sqrt{\kappa^2+\tau^2},$$

*with curvature $\kappa$ and complex torsion $\tau$ of $\gamma$.*

(2) *Conversely, given positive constants $\kappa$, $\tau$ and a constant $\mu$ with $|\mu| \leqq \kappa/\sqrt{\kappa^2+\tau^2}$, we have a unique Killing ordinary helix on $M_n(c)$ up to the action of holomorphic isometries of $M_n(c)$ whose curvature is $\kappa$, torsion is $\tau$, and the first complex torsion $\tau_{12}$ is $\mu$.*

(3) *If* $|\mu| > \kappa/\sqrt{\kappa^2+\tau^2}$, *we have no such a Killing ordinary helix on* $M_n(c)$.

**Theorem B** ([MA]) *Let* $M_2(c)$ *be a 2-dimensional nonflat complex space form. Then the following hold*:

(1) *All complex torsions of each Killing ordinary helix* $\gamma$ *in* $M_2(c)$ *are constant. They are expressed as either*

$$\tau_{12} = \kappa/\sqrt{\kappa^2+\tau^2}, \quad \tau_{13} = 0, \quad \tau_{23} = \tau/\sqrt{\kappa^2+\tau^2} \tag{2.2}$$

*or*

$$\tau_{12} = -\kappa/\sqrt{\kappa^2+\tau^2}, \quad \tau_{13} = 0, \quad \tau_{23} = -\tau/\sqrt{\kappa^2+\tau^2} \tag{2.3}$$

*by use of curvature* $\kappa$ *and torsion* $\tau$ *of* $\gamma$.

(2) *Conversely, for given positive constants* $\kappa$, $\tau$ *we have two Killing ordinary helices on* $M_2(c)$ *whose curvature is* $\kappa$ *and torsion is* $\tau$ *up to the action of holomorphic isometries on* $M_2(c)$. *Their complex torsions correspond to either* (2.2) *or* (2.3).

It is well-known that every isometry of a nonflat complex space form $M_n(c)$ is either holomorphic or anti-holomorphic with respect to the given complex structure $J$ of $M_n(c)$. We should note that those Killing ordinary helices having the same curvature $\kappa$ and torsion $\tau$ with complex torsions (2.2) or (2.3) are mutually congruent by some anti-holomorphic isometry of $M_n(c)$.

Similarly those statements in Proposition 2 do not hold for ordinary helices on other symmetric spaces of rank one, which are a quaternionic projective space $\mathbb{H}P^n$, a quaternionic hyperbolic space $\mathbb{H}H^n$, a Cayley plane $\mathbb{O}P^2$ and a Cayley hyperbolic plane $\mathbb{O}H^2$. We denote by $\mathbb{K}$ either the field of quaternion numbers $\mathbb{H}$ or the Cayley algebra $\mathbb{O}$. For an ordinary helix $\gamma$ on $\mathbb{K}P^n$ or $\mathbb{K}H^n$ we put $\tau_{ij}(s) = \|\mathrm{Proj}_j(V_i(s))\|$, $1 \leqq i < j \leqq 3$, where $\mathrm{Proj}_j$ denotes the projection of the tangent space at $\gamma(s)$ onto the $\mathbb{K}$-subspace generated by $V_j(s)$. These correspond to the absolute value of the complex torsions for ordinary helices on nonflat complex space forms. On these spaces $\mathbb{K}P^n$ and $\mathbb{K}H^n$ every isometry $\varphi$ preserves the $\mathbb{K}$-structure; that is, $d\varphi$ maps the $\mathbb{K}$-subspace spanned by a tangent vector $v$ to the $\mathbb{K}$-subspace spanned by $d\varphi(v)$. Therefore if an ordinary helix $\gamma$ on $\mathbb{K}P^n$ or $\mathbb{K}H^n$ is generated by some Killing vector field on such a space, then $\tau_{ij}$ should be constant along $\gamma$. We now take for example an ordinary helix of curvature $\kappa$

and of torsion $\tau$ on $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with $\tau_{13}(0) \neq 0$. Then $\tau_{12}$ is not constant, because $\tau_{12}'(s) = \tau \cdot \tau_{13}(s)$. Hence it follows from the fact that $\mathbb{K}P^n$ admits $\mathbb{C}P^2$ as a totally geodesic submanifold and $\mathbb{K}H^n$ admits $\mathbb{C}H^2$ as a totally geodesic submanifold and Theorem A that the following result holds.

**Proposition 3** *Let $M$ be a Riemannian symmetric space of rank one which is not a real space form, and $(\kappa, \tau)$ be a pair of positive numbers.*

(1) *There exist on $M$ infinitely many congruency classes of ordinary helices which are not Killing ordinary helices whose curvature is $\kappa$ and whose torsion is $\tau$.*

(2) *When $M$ is not a plane, that is, $M \neq \mathbb{C}P^2$, $\mathbb{H}P^2$, $\mathbb{O}P^2$, $\mathbb{C}H^2$, $\mathbb{H}H^2$, $\mathbb{O}H^2$, there exist on $M$ infinitely many congruency classes of Killing ordinary helices whose curvature is $\kappa$ and whose torsion is $\tau$.*

## 3. Construction of closed ordinary helices with self-intersections in $\mathbb{C}P^2$

In this section we construct a closed ordinary helix with self-intersection points on a complex projective plane $\mathbb{C}P^2$ by using Riemannian submanifold theory. We consider a quotient of a 2-dimensional flat torus $N = (S^1 \times S^1)/\phi$. Here representing the first component by $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the second component by $S^1 = \{(a_1, a_2) \in \mathbb{R}^2 \mid (a_1)^2 + (a_2)^2 = 1\}$, we define the identification $\phi$ by $\phi\big((e^{i\theta}, (a_1, a_2))\big) = (-e^{i\theta}, (-a_1, -a_2))$. The Riemannian metric on $N$ is induced by the metric on a torus $S^1 \times S^1$ which is given by $\langle A+\xi, B+\eta\rangle = \frac{2}{9}\langle A, B\rangle_{S^1} + \frac{2}{3}\langle \xi, \eta\rangle_{S^1}$ for tangent vectors $A, B \in TS^1$ of the first component and tangent vectors $\xi, \eta \in TS^1$ of the second component, where $\langle\ ,\ \rangle_{S^1}$ denotes the canonical metric on $S^1$.

In order to obtain an ordinary helix with self-intersection points we make use of the following isometric embedding $f : N \to \mathbb{C}P^2(4)$ defined by

$$f\big([e^{i\theta}, (a_1, a_2)]\big) = \pi\left(\frac{1}{3}\big(e^{-\frac{2i\theta}{3}} + 2a_1 e^{\frac{i\theta}{3}}\big), \frac{\sqrt{2}}{3}\big(e^{-\frac{2i\theta}{3}} - a_1 e^{\frac{i\theta}{3}}\big), \frac{2}{\sqrt{6}} i a_2 e^{\frac{i\theta}{3}}\right), \quad (3.1)$$

where $\pi : S^5(1) \to \mathbb{C}P^2(4)$ is the Hopf fibration. By virtue of the result in [AM], the image $f \circ \gamma$ of a circle $\gamma$ of curvature $1/2$ on $N$ is an ordinary helix on $\mathbb{C}P^2(4)$ with curvature $\sqrt{3}/2$ and torsion $\sqrt{3/2}$. The complex torsions of $f \circ \gamma$ are described as

$$\tau_{12} = \sqrt{\frac{2}{3}}\cos 3\Big(\frac{1}{2}s + \psi_0\Big), \quad \tau_{13} = -\sin 3\Big(\frac{1}{2}s + \psi_0\Big),$$
$$\tau_{23} = -\frac{1}{\sqrt{3}}\cos 3\Big(\frac{1}{2}s + \psi_0\Big).$$

Here, $\psi_0$ is the angle between $\dot{\gamma}(0)$ and the unit vector $u$ tangent to the first component of $N$. As the complex torsions of $f \circ \gamma$ are not constants, we find it is not an integral curve of any Killing vector field on $\mathbb{C}P^2(4)$.

We now consider the universal Riemannian covering $p : \mathbb{R}^2 \to N$. Regarding the Riemannian metric on $N$, we can choose a fundamental region for $N$ in $\mathbb{R}^2$ as $\mathfrak{F} = [0, 2\sqrt{2}\pi/3) \times [0, \sqrt{6}\pi/3)$. Two points $(x_1, x_2)$ and $(y_1, y_2)$ on $\mathbb{R}^2$ satisfy $p((x_1, x_2)) = p((y_1, y_2))$ if and only if they satisfy one of the following conditions:

i) $x_1 - y_1 = 2\sqrt{2}m_1\pi/3$, $x_2 - y_2 = 2\sqrt{6}m_2\pi/3$ for some $m_1, m_2 \in \mathbb{Z}$,

ii) $x_1 - y_1 = \sqrt{2}(2m_1 + 1)\pi/3$, $x_2 - y_2 = \sqrt{6}(2m_2 + 1)\pi/3$ for some $m_1, m_2 \in \mathbb{Z}$.

Let $\tilde{\gamma}$ denote a covering circle in $\mathbb{R}^2$ of a circle $\gamma$ of curvature $1/2$ on $N$. Then the curve $\tilde{\gamma}$ is a circle of radius 2 in the sense of Euclidean Geometry. This, together with the fact that $f$ is an isometric imbedding, shows that the curve $f \circ \gamma$ is a closed curve of length $4\pi$ in $\mathbb{C}P^2(4)$. Moreover, since the curvature $1/2$ of the circle $\gamma$ is less than $3/(\sqrt{2}\pi)$, it has self-intersection points and hence so does the ordinary helix $f \circ \gamma$. Suppose $\gamma(s_0)$ is a self-intersection point. Denoting the tangential vector $\dot{\gamma}(s_0)$ by $(v_1, v_2) \in \mathbb{R}^2 \cong T_{\gamma(s_0)}N$, we have

$$\tilde{\gamma}(s) = 2\begin{pmatrix} v_1 \sin\dfrac{s - s_0}{2} + v_2\Big(\cos\dfrac{s - s_0}{2} - 1\Big) \\ v_2 \sin\dfrac{s - s_0}{2} - v_1\Big(\cos\dfrac{s - s_0}{2} - 1\Big) \end{pmatrix} + \tilde{\gamma}(s_0).$$

If $\gamma(s_0 + s_1) = \gamma(s_0)$, then we get in the case i) that

$$\begin{cases} v_1 \sin\dfrac{s_1}{2} + v_2\Big(\cos\dfrac{s_1}{2} - 1\Big) = \dfrac{\sqrt{2}}{3}m_1\pi, \\ v_2 \sin\dfrac{s_1}{2} - v_1\Big(\cos\dfrac{s_1}{2} - 1\Big) = \dfrac{\sqrt{6}}{3}m_2\pi. \end{cases}$$

As $v_1^2 + v_2^2 = 1$, we obtain in this case that

$$\begin{cases} \sin\dfrac{s_1}{2} = \dfrac{\sqrt{2}}{3}\pi(m_1v_1 + \sqrt{3}m_2v_2), \\ \cos\dfrac{s_1}{2} = \dfrac{\sqrt{2}}{3}\pi(m_1v_2 - \sqrt{3}m_2v_1) + 1. \end{cases}$$

Thus we find $(v_1, v_2)$ satisfies either

I) $\pi(m_1^2 + 3m_2^2) = 3(-\sqrt{2}v_2m_1 + \sqrt{6}v_1m_2)$, or

II) $\pi\{(2m_1+1)^2 + 3(2m_2+1)^2\} = 3\{-\sqrt{2}v_2(2m_1+1) + \sqrt{6}v_1(2m_2+1)\}$

for some pair of intergers $(m_1, m_2) \neq (0,0)$ corresponding to the conditions i) and ii). In our case, as the curvature $1/2$ of $\gamma$ is greater than $3/(\sqrt{6}\pi)$, such conditions might occur only for $(m_1, m_2) = (\pm 1, 0)$ for the case I) and for $(m_1, m_2) = (0,-1), (0,0), (-1,0), (-1,-1), (1,-1), (1,0)$ for the case II) (see Figure 1). Checking these conditions (or looking Figure 1 carefully), we find $\gamma$ does not have self-intersection points corresponding to $(m_1, m_2) = (1,-1), (1,0)$ for II), and has self-intersection points corresponding to other conditions. Thus we find $\gamma$ has 6 self-intersection points, and so does $f \circ \gamma$.

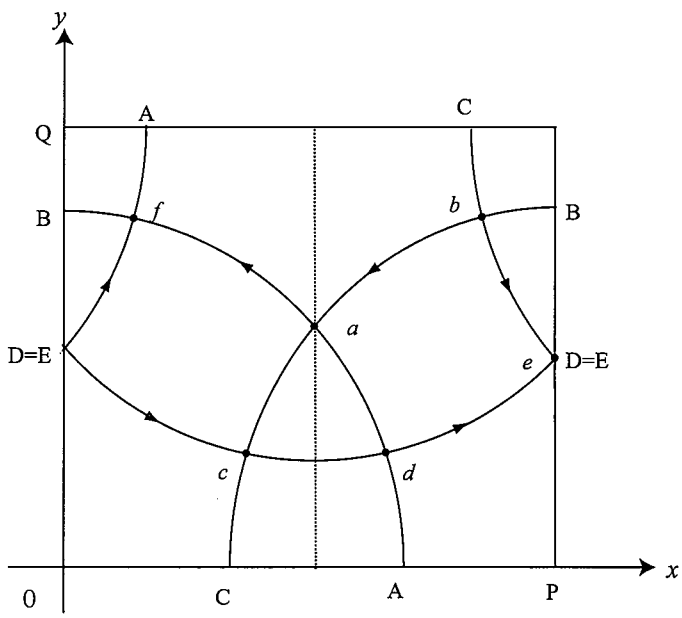


Fig. 1. Circle of curvature $1/2$ on $N$
(P$(2\sqrt{2}\pi/3, 0)$, Q$(0, \sqrt{6}\pi/3)$, A$(2,0)$, B$(0,2)$).

Summarizing up we obtain the following.

**Theorem 1** *Let $f : N \longrightarrow \mathbb{C}P^2(4)$ denote the isometric embedding defined by (3.1) and $\gamma$ be a circle of curvature $1/2$ on $N$. Then the curve $f \circ \gamma$ satisfies the following:*

(1) *It is a closed ordinary helix of length $4\pi$ with curvature $\sqrt{3}/2$ and torsion $\sqrt{3/2}$ on $\mathbb{C}P^2(4)$.*

(2) *It is not generated by any Killing vector field on $\mathbb{C}P^2(4)$ and has 6 self-intersection points.*

**Remark 1** In connection with submanifolds we have the following Killing ordinary helix on $\mathbb{C}P^2(4)$. Let $h : \mathbb{C}P^1(2)\ (= S^2(2)) \to \mathbb{C}P^2(4)$ denote the second Veronese embedding given by $h([z_0, z_1]) = [z_0^2, \sqrt{2}z_0z_1, z_1^2]$ with homogeneous coordinate $(z_0, z_1)$ on $\mathbb{C}P^1(2)$. For a circle of curvature $1/\sqrt{2}$ on $\mathbb{C}P^1(2)$, the curve $h \circ \gamma$ is a Killing ordinary helix of curvature $\sqrt{3/2}$ and torsion $\sqrt{3}$. This ordinary helix is closed with length $2\sqrt{2/5}\pi$ and does not have self-intersection points (see [AM]).

## 4. Characterization of real space forms by ordinary helices

A Riemannian manifold $M$ is said to be *homogeneous* if the isometry group $G$ of $M$ acts transitively on $M$. Our study in this section is motivated by the following theorem:

**Theorem C** ([MT]) *Let $M$ be a Riemannian homogeneous space and $\kappa$ be a positive constant. If every circle of curvature $\kappa$ on $M$ is generated by some Killing vector field on $M$, then $M$ is either a Euclidean space or a Riemannian globally symmetric space of rank one.*

Our aim here is to provide the following characterization of real space forms by ordinary helices in the class of Riemannian homogeneous spaces.

**Theorem 2** *Let $M$ be a Riemannian homogeneous space and $\kappa$, $\tau$ be positive constants. Every ordinary helix of curvature $\kappa$ and torsion $\tau$ on $M$ is generated by some Killing vector field on $M$ if and only if $M$ is a real space form.*

Let $K$ denote the compact isotropy subgroup of $G$ such as $M = G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. Take an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. For each $X \in \mathfrak{p}$, we define an element $\Lambda_{\mathfrak{p}}(X) \in \mathfrak{so}(\mathfrak{p})$ by

$$\Lambda_{\mathfrak{p}}(X)(Y) = \frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y)$$

for $Y \in \mathfrak{p}$, where $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is given by the equality

$$\langle U(X,Y), Z\rangle = \frac{1}{2}\{\langle [Z,X]_{\mathfrak{p}}, Y\rangle + \langle [Z,Y]_{\mathfrak{p}}, X\rangle\}$$

for every $X, Y, Z \in \mathfrak{p}$. We define a linear map $\Lambda : \mathfrak{g} \to \mathfrak{so}(\mathfrak{p})$ by $\Lambda(X) = \mathrm{ad}_{X_{\mathfrak{k}}} + \Lambda_{\mathfrak{p}}(X_{\mathfrak{p}})$, where for $X \in \mathfrak{g}$ we denote by $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ with $X_{\mathfrak{k}} \in \mathfrak{k}$ and $X_{\mathfrak{p}} \in \mathfrak{p}$. We denote the projection by $\varpi : G \to G/K$. Along the same lines as in [MT] we can conclude the following.

**Proposition 4** *Let $\{X, Y, Z\}$ be a triplet of orthonormal vectors in $\mathfrak{p}$ and $H$ be an element of $\mathfrak{k}$. The orbit $\gamma(t) = \varpi(\exp t(H + X))$ is an ordinary helix on $G/K$ of curvature $\kappa$ and torsion $\tau$ with initial frame $(X, Y, Z)$ if and only if the following equalities hold*:

$$\begin{aligned}
\Lambda(H+X)(X) &= [H,X] + \Lambda_{\mathfrak{p}}(X)(X) = \quad \kappa Y,\\
\Lambda(H+X)(Y) &= [H,Y] + \Lambda_{\mathfrak{p}}(X)(Y) = -\kappa X + \tau Z,\\
\Lambda(H+X)(Z) &= [H,Z] + \Lambda_{\mathfrak{p}}(X)(Z) = \quad -\tau Y.
\end{aligned}$$

**Corollary** *Let $M = G/K$ be a Riemannian symmetirc space. Let $\{X, Y, Z\}$ be a triplet of orthonormal vectors in $\mathfrak{p}$ and $H$ be an element of $\mathfrak{k}$. The orbit $\gamma(t) = \varpi(\exp t(H + X))$ is an ordinary helix on $G/K$ of curvature $\kappa$ and torsion $\tau$ with initial frame $(X, Y, Z)$ if and only if the following equalities hold*:

$$[H,X] = \kappa Y, \quad [H,Y] = -\kappa X + \tau Z, \quad [H,Z] = -\tau Y.$$

*Proof of Theorem* 2. The "if" part of the theorem is Proposition 2. In the following we prove the "only if" part. Suppose that every ordinary helix with curvature $\kappa$ and torsion $\tau$ in $M = G/K$ is an orbit of one-parameter subgroup of $G$. Let $\{X, Y, Z\}$ be an arbitrary triplet of mutually orthonormal vectors in $\mathfrak{p}$. Take two ordinary helices $\gamma_1$ and $\gamma_2$ of curvature $\kappa$ and torsion $\tau$ which go through the point $eK(= \gamma_1(0) = \gamma_2(0))$ and whose initial frames $(V_1(0), V_2(0), V_3(0))$ are $(X, Y, Z)$ and $(X, -Y, Z)$, respectively. By Proposition 4 we have $H_1, H_2 \in \mathfrak{k}$ with

$$[H_1, X] + \Lambda_{\mathfrak{p}}(X)(X) = \kappa Y, \quad [H_2, X] + \Lambda_{\mathfrak{p}}(X)(X) = -\kappa Y.$$

We then obtain $[H, X] = Y$ for $H = (1/2\kappa)(H_1 - H_2)$. Repeating the same discussion as in the proof of Theorem 3.1 in [MT], we find that the linear

isotropy group $\mathrm{Ad}(K)$ acts transitively on the unit sphere $S(\mathfrak{p})$ in $\mathfrak{p}$. Hence $M$ is a Euclidean space or a Riemannian globally symmetric space of rank one.

On each Riemannian symmetric space of rank one which is not a real space form we have by Proposition 3 ordinary helices of curvature $\kappa$ and torsion $\tau$ which are not generated by any Killing vector field on this space. We hence get our conclusion. □

Before closing this paper, we write down the form of an ordinary helix on a symmetric space $G/K$. For a curve $\gamma : \mathbb{R} \to G/K$ we represent its framing $F_\gamma : \mathbb{R} \to G$, which is a map with $\gamma = \varpi \circ F_\gamma$, as $F_\gamma = g_\gamma h_\gamma$ by using maps $g_\gamma : \mathbb{R} \to G$, $h_\gamma : \mathbb{R} \to K$ with the property that $g_\gamma^{-1}(dg_\gamma/dt) \in \mathfrak{m}$.

**Proposition 5** *A smooth curve $\gamma$ on a Riemannian symmetric space $G/K$ is an ordinary helix of curvature $\kappa$ and torsion $\tau$ with initial frame $(X, Y, Z)$ if and only if $g_\gamma$ is a solution of the following differential equation with $g_\gamma(0) = I$:*

$$g_\gamma^{-1}\frac{dg_\gamma}{dt} = \frac{1}{\kappa^2+\tau^2}\left(\kappa^2 \cos\sqrt{\kappa^2+\tau^2}t + \tau^2\right)X + \frac{\kappa \sin\sqrt{\kappa^2+\tau^2}t}{\sqrt{\kappa^2+\tau^2}}Y + \frac{\kappa\tau}{\kappa^2+\tau^2}\left(1 - \cos\sqrt{\kappa^2+\tau^2}t\right)Z. \tag{4.1}$$

*In this case, each entry of the solution $g_\gamma(t)$ is represented by the power series of $t$ with infinite radius of convergence.*

*Proof.* Since we can easily get the last claim by standard argument (cf. [AMU]), we only show the "only if" part. Suppose $\gamma$ is an ordinary helix. Let $F_\gamma^{-1}(dF_\gamma/dt) = \alpha_\mathfrak{k} + \alpha_\mathfrak{m}$ be the decomposition according as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. By using $g_\gamma$ and $h_\gamma$ we have

$$F_\gamma^{-1}(dF_\gamma/dt) = h_\gamma^{-1}(dh_\gamma/dt) + \mathrm{Ad}\, h_\gamma^{-1}(g_\gamma^{-1}(dg_\gamma/dt)).$$

We here note that $\alpha_\mathfrak{k} = h_\gamma^{-1}(dh_\gamma/dt)$ always has a solution for a given $\alpha_\mathfrak{k}$ (see [KN]). We then have $\alpha_\mathfrak{m} = \mathrm{Ad}\, h_\gamma^{-1}(g_\gamma^{-1}(dg_\gamma/dt))$ for such a solution $h_\gamma$.

By the equation (2.1) we find the curve $\widetilde{\alpha}_\mathfrak{m}$ which is defined by $\mathrm{Ad}\, h_\gamma \alpha_\mathfrak{m}$ satisfies

$$\frac{d^3\widetilde{\alpha}_\mathfrak{m}}{dt^3} = -(\kappa^2+\tau^2)\frac{d\widetilde{\alpha}_\mathfrak{m}}{dt},$$

hence we see by initial condition $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(0) = -\kappa^2 X + \kappa\tau Z$ that

$$\frac{d^2\widetilde{\alpha}_{\mathfrak{m}}}{dt^2} = -(\kappa^2+\tau^2)\widetilde{\alpha}_{\mathfrak{m}} + \tau^2 X + \kappa\tau Z.$$

If we set

$$\widehat{\alpha}_{\mathfrak{m}} = \widetilde{\alpha}_{\mathfrak{m}} - \frac{1}{\kappa^2+\tau^2}\left(\tau^2 X + \kappa\tau Z\right),$$

then we have $d^2\widehat{\alpha}_{\mathfrak{m}}/dt^2 = -(\kappa^2+\tau^2)\widehat{\alpha}_{\mathfrak{m}}$. Solving this equation, we obtain

$$\begin{aligned}\widetilde{\alpha}_{\mathfrak{m}} = \cos\sqrt{\kappa^2+\tau^2}t\Big(X - \frac{1}{\kappa^2+\tau^2}(\tau^2 X + \kappa\tau Z)\Big)\\ + \frac{\kappa\sin\sqrt{\kappa^2+\tau^2}t}{\sqrt{\kappa^2+\tau^2}}Y + \frac{1}{\kappa^2+\tau^2}(\tau^2 X + \kappa\tau Z),\end{aligned}$$

which lead us to the desirable differential equation because $\widetilde{\alpha}_{\mathfrak{m}} = g_\gamma^{-1}(dg_\gamma/dt)$. □

This proposition also shows Corollary. For an orbit $\gamma(t) = \varpi(\exp t(H+X))$ on a symmetric space $G/K$ we may have $g_\gamma(t) = \exp(t(H+X))\cdot\exp(-tH)$, hence obtain $g_\gamma^{-1}(dg_\gamma/dt) = \mathrm{Ad}\exp(tH)X$. This shows that the curve $\widetilde{\alpha}_{\mathfrak{m}} = g_\gamma^{-1}(dg_\gamma/dt)$ satisfies $\left(d^j\widetilde{\alpha}_{\mathfrak{m}}/dt^j\right)(0) = (\mathrm{ad}\,H)^j X$. On the other hand, by (4.1) we have

$$\frac{d\widetilde{\alpha}_{\mathfrak{m}}}{dt}(0) = \kappa Y,\quad \frac{d^2\widetilde{\alpha}_{\mathfrak{m}}}{dt^2}(0) = -\kappa^2 X + \kappa\tau Z,\quad \frac{d^3\widetilde{\alpha}_{\mathfrak{m}}}{dt^3}(0) = -\kappa(\kappa^2+\tau^2)Y.$$

Thus we get Corollary.

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Toshiaki Adachi
Department of Mathematics
Nagoya Institute of Technology
Nagoya 466-8555, Japan
E-mail: adachi@nitech.ac.jp

Sadahiro Maeda
Department of Mathematics
Shimane University
Matsue, Shimane 690-8504, Japan
E-mail: smaeda@math.shimane-u.ac.jp

Seiichi Udagawa
Department of Mathematics
School of Medicine
Nihon University
Itabashi, Tokyo 173-0032, Japan
E-mail: sudagawa@med.nihon-u.ac.jp