Harmonic maps from a two-torus into a complex projective space

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Introduction

This paper is an exposition of recent results of Burstall’s in [B] which says that any non-superminimal harmonic map from a two-torus into a complex projective space is covered by a primitive map of finite type into a certain generalized flag manifold. In [B-P], the outline of the proof of his result is appeared. First, it is observed that a primitive map \( \psi \) from a two-torus \( T^2 \) into a \( k \)-symmetric space \( G/K \) is of finite type if, for some (hence every) framing of \( \psi \), \( \alpha_m'(\partial/\partial z) \) is semisimple on a dense subset of \( T^2 \), where \( \alpha_m' \) is defined as follows: For some (local) lifting \( F : T^2 \to G \) of \( \psi : T^2 \to G/K \), set \( \alpha = F^{-1}dF \), which is the pull-back of the Maurer-Cartan form of \( G \). Corresponding to the reductive decomposition \( g = k + m \), where \( m = T_0(G/K) \), set \( \alpha = \alpha_m + \alpha_k \), and \( \alpha_m = \alpha_m' + \alpha''_m \) is a decomposition into \((1,0)\)-form and \((0,1)\)-form, respectively.

In this paper, we give a proof of the fact that any non-superminimal harmonic map \( \varphi \) from a two-torus into a complex projective space may be lifted to a primitive map with semisimple \( \alpha_m' \) into a certain generalized flag manifold, where the twistor space is chosen according to the isotropy order of \( \varphi \).

1. Generalized flag manifold associated to a non-superminimal harmonic map into a complex projective space

Let \( \varphi : S \to \mathbb{C}P^n \) be a harmonic map, where \( \mathbb{C}P^n = SU(n+1)/S(U(1) \times U(n)) \). Let \( L_0 \) be the pull-back of universal bundle over \( \mathbb{C}P^n \) by \( \varphi \). \( L_0 \) is a subbundle of the trivial bundle \( V(\mathbb{C}^{n+1}) = S \times \mathbb{C}^{n+1} \). We equip \( V(\mathbb{C}^{n+1}) \) with the standard Hermitian connected structure with Hermitian metric \( \langle , \rangle \) given by

\[
\langle f, g \rangle = \sum_{i=0}^n f_i \overline{g_i}, \quad \text{for } f = (f_0, f_1, \cdots, f_n), \ g = (g_0, g_1, \cdots, g_n)
\]

For any subbundle \( F \) of \( V(\mathbb{C}^{n+1}) \), we denote by \( F^\perp \) the Hermitian orthogonal complement of \( F \) in \( V(\mathbb{C}^{n+1}) \). Then, \( F \) and \( F^\perp \) are both equipped with the induced Hermitian connected structures from \( V(\mathbb{C}^{n+1}) \). Moreover, \( F \) and \( F^\perp \) both have the Koszul-Malgrange holomorphic structures. Let \( A_e^{F,F^\perp} \) be the \((1,0)\)-part of the second fundamental form of \( F \) in \( V(\mathbb{C}^{n+1}) \). By taking the image of the second fundamental form, we may define the new subbundle of \( V(\mathbb{C}^{n+1}) \), which is extended to smooth subbundle over \( S \) (see [B-P-W]).
Now, starting from \( L_0 \), we may have the harmonic sequence \( L_0 \rightarrow L_1 \rightarrow \cdots L_{r-1} \rightarrow R \), where \( L_i = \text{Im} A_{L_i-1, L_i-1} \) for \( i = 1, \cdots, r-1 \) and \( R = V(C^{n+1}) \oplus (\bigoplus_{i=0}^{r-1} L_i) \). This situation means that each of \( L_0, L_1, \cdots, L_{r-1} \) and \( R \) are orthogonal to each other with respect to the Hermitian metric on \( V(C^{n+1}) \). In this case, we say that \( \varphi \) has \( \partial' \)-isotropy order \( r \). From the definition of harmonic sequence, it is always true that \( r \geq 1 \). For notational simplicity, set \( L_r = R \). Set \( G = SU(n+1) \). Fix any point \( p \in S \) and define \( Q \in G \) by

\[
Q = \zeta^i \quad \text{on} \quad (L_i)_p \quad \text{for} \quad i = 0, \cdots, r
\]

where \( \zeta = \exp(2\pi i/r + 1) \). Then, \( \tau = AdQ \) is an order \((r+1)\)-automorphism of \( G \) and the identity component of its fixed set is \( S(U(1) \times \cdots \times U(1) \times U(n+1-r)) \), which we denote by \( K \). Hence, we define a map \( \psi : S \rightarrow N = G/K \) by

\[
\psi(q) = ((L_0)_q, (L_1)_q, \cdots, (L_r)_q), \quad \text{for} \quad q \in S
\]

Choose the base point \( o = \psi(p) \). The complexification \( g^C \) of Lie algebra \( g \) of \( G \) is decomposed into the eigenspaces of \( \tau \):

\[
g^C = \sum_{j \in \mathbb{Z}_{r+1}} g_j
\]

where

\[
m^C = \sum_{j=1}^{r} g_j, \quad k^C = g_0
\]

For \( x = g \cdot o \in N \), define \( \hat{\tau} : N \rightarrow N \) by \( \hat{\tau}(g \cdot o) = \tau(g) \cdot o \). Define \( \hat{\tau}_x : N \rightarrow N \) by \( \hat{\tau}_x = g \circ \hat{\tau} \circ g^{-1} \). We may use the Killing form of \( g \) to equip \( N \) with a metric for which each of the \( \hat{\tau}_x \) is an isometry so that \( N \) has the structure of an \((r+1)\)-symmetric space ([K]). Let \([g_i]\) be a subbundle of \( N \times g \) of which the fibre at \( x \in N \) is given by \([g_i]_x = Adgg_i\). Then, \([g_i]_x \) is \( \zeta^1 \)-eigenspace of \( d\hat{\tau}_x \). For \( x = g \cdot o \in N \), the map \( g \rightarrow T_x N \) given by

\[
\xi \rightarrow \frac{d}{dt} |_{t=0} \exp t \xi \cdot x
\]

restricts to an isomorphism \( Adgm \cong T_o N \). The inverse map \( \beta_x : T_x N \rightarrow Adgm \subset g \) may be viewed as a \( g \)-valued 1-form \( \beta \) on \( N \), which is called Maurer-Cartan form for \( N \) (see [B-R]).

**Definition.** A map \( \psi : S \rightarrow G/K \) of a Riemann surface is called primitive if \( \psi^* \beta(\partial/\partial z) \) takes values in \([g_1]\), or equivalently \( \alpha'_m(\partial/\partial z) \) takes values in \( g_1 \) for any framing \( F : S \rightarrow G \).

**Lemma 1.1.** \( A^{(L_i)}_{L_i} \) is \( g_1 \)-valued for \( i = 0, \cdots, r \), where \( L_{r+1} = L_0 \). Moreover, \( \psi \) is a primitive map.

**Proof.** Let \( f_i \) be a local section of \( L_i \) for \( i = 0, \cdots, r \). If we set \( A_{L_i}^{L_i+1} (f_i) = a_{i,i+1} f_{i+1} \), then we have, at \( p \)

\[
(AdQ(A_{L_i}^{L_i+1}))(Qf_i) = Q(a_{i,i+1} f_{i+1}) = \zeta^i A_{L_i}^{L_i+1}
\]
Therefore, we see that $AdQ(A_{z}^{L_{i},L_{i+1}}) = \zeta A_{z}^{L_{i},L_{i+1}}$. Next, we show the second assertion. From [B-R], we see that $\psi^*\beta(\partial/\partial z)$ is the sum of the second fundamental forms of $L_i$'s $(i = 0, \cdots, r)$. Then, from the construction of harmonic sequence, it follows that

\begin{equation}
\psi^*\beta(\partial/\partial z) = \sum_{i=0}^{r} A_{z}^{L_{i},L_{i+1}}
\end{equation}

where $L_{r+1} = L_0$ for convention. Since $\psi^*\beta_p = \alpha_m$, it follows from (1.1) that $\psi$ is a primitive map. q.e.d.

It remains to show that $\alpha'_m$ is a semisimple element on a dense subset of $S$.

2. Non-conformal harmonic maps into $\mathbb{C}P^n$.

Let $f_0$ be a local non-zero holomorphic section of $L_0$ and $(f_1, \cdots, f_n)$ a local field of unitary frames of $L_1 = L_0^\perp$. If we set $f_i = det(f_0, \cdots, f_n)^{-\frac{1}{n+1}}f_i$ for $i = 0, \cdots, n$, then $det(f_0, \cdots, f_n) = 1$. Moreover, we see that $|f_0| \cdots |f_n| = 1$. Now, define $F_i$ by $F_i = \exp(-w_i)f_i$ for $i = 0, \cdots, n$, where $w_i = \log |f_i|$. Then, $F = (F_0, F_1, \cdots, F_n)$ is $SU(n+1)$-valued locally defined function on $S$. Let $g = su(n+1)$ be a Lie algebra of $SU(n+1)$. Then, $g^C = sl(n+1, \mathbb{C})$ is decomposed as

\begin{equation}
g^C = g_0 \oplus g_1
\end{equation}

where $g_i$ is the $\zeta^i$-eigenspace of $\tau$ for $i = 0, 1$. We see that $g_1$ is isomorphic to the complexification of the tangent space at the base point and that $A_{z}^{L_0,L_0^\perp}$ and $A_{z}^{L_0^\perp,L_0}$ are $g_1$-valued (c.f. Lemma 1.1). Write

\begin{equation}
A_{z}^{L_0,L_0^\perp}(f_0) = \sum_{i=1}^{n} a_i f_i, \quad A_{z}^{L_0^\perp,L_0}(f_0) = \sum_{i=1}^{n} b_i f_i
\end{equation}

Let $g = m + k$ be a Cartan decomposition. Then, $m^C = g_1$. Let $\alpha'_m(\partial/\partial z)$ be a $m$-part of $F^{-1}(\partial F/\partial z)$. Then, for $j = 1, \cdots, n$

\begin{equation}
(F^{-1}(\partial F/\partial z))_{j,0} = < A_{z}^{L_0,L_0^\perp}(F_0), F_j > = a_j w_{j,0},
\end{equation}

\begin{equation}
(F^{-1}(\partial F/\partial z))_{0,j} = -< F^{-1}(\partial F/\partial z), F_j > = -\overline{b_j} w_{j,0},
\end{equation}

where $w_{j,0} = \exp(w_j - w_0)$. Thus, we obtain

\begin{equation}
\alpha'_m(\partial/\partial z) = \begin{pmatrix}
0 & -\overline{\mathbf{b}} \\
\mathbf{a} & 0_n
\end{pmatrix}
\end{equation}

where $\mathbf{a} = (a_1 w_{1,0}, a_2 w_{2,0}, \cdots, a_n w_{n,0})$, $\mathbf{b} = (b_1 w_{1,0}, b_2 w_{2,0}, \cdots, b_n w_{n,0})$ and $0_n$ is an $n \times n$-zero matrix. If we write $\langle \mathbf{a}, \overline{\mathbf{b}} \rangle = r \cdot \exp(r\theta)$ for non-zero real numbers $r, \theta$, then the eigenvalues of $\alpha'_m(\partial/\partial z)$ are 0 (with multiplicity $n - 1$) and $\pm \sqrt{r} \exp(i\theta/2)$. If we set $A = A_{z}^{L_0^\perp,L_0} \circ A_{z}^{L_0,L_0^\perp}$, then we see that $trace A = \langle \mathbf{a}, \overline{\mathbf{b}} \rangle$. Therefore, $\varphi : S \rightarrow \mathbb{C}P^n$ is non-conformal at $x \in S$ if and only if $\alpha'_m(\partial/\partial z)$ is semisimple at $x \in S$. If $S$ is a two-torus, $trace A$ is a constant, hence we have proved
Proposition 2.1. For any non-conformal harmonic map $T^2 \to \mathbb{C}P^n$, $\alpha'_m(\partial/\partial z)$ is a semisimple element.

Thus, we have

Theorem 2.2 [B-F-P-P]. Any non-conformal harmonic map from $T^2$ into $\mathbb{C}P^n$ is of finite type.

3. Weakly conformal harmonic map $S \to \mathbb{C}P^n$.

Let $\varphi : S \to \mathbb{C}P^n$ be a weakly conformal harmonic map with isotropy order $r (\geq 1)$.

Let $\hat{f}_i$ be a local non-zero holomorphic section of $L_i$ for $i = 0, \ldots, r - 1$ and $(\hat{f}_r, \ldots, \hat{f}_n)$ be a local field of unitary frame for $L_r$. We may suppose, without loss of generality, that

$$A_z^{L_i, L_{i+1}}(f_i) = f_{i+1} \quad \text{for} \quad i = 0, \ldots, r - 2.$$ 

Set $\hat{f} = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_n)$. We reset $f_i = (\det \hat{f})^{-\frac{1}{n-r}} \hat{f}_i$ for $i = 0, \ldots, n$. For $i = 0, \ldots, n$, define $F_i$ by $F_i = \exp(-w_i) f_i$, where $w_i = \log | f_i |$. Then, $F = (F_0, F_1, \ldots, F_n)$ is $SU(n+1)$-valued. Set

$$A_z^{L_{r-1}, L_r}(f_{r-1}) = \sum_{j=r}^{n} a_j f_j, \quad A_z^{L_0, L_r}(f_0) = \sum_{j=r}^{n} b_j f_j.$$ 

Then, we have

$$\left(IF^{-1} \partial F/\partial z\right)_{j,0} = <\partial F_0/\partial z, \overline{F}_j> = \exp(w_1 - w_0) \delta_{1,j}, \quad (j \geq 1),$$

$$\left(IF^{-1} \partial F/\partial z\right)_{i,j} = <\partial F_j/\partial z, \overline{F}_i>$$

$$= <A_z^{L_j, L_{j+1}}(F_j), \overline{F}_i> = \exp(w_{j+1} - w_j) \delta_{i,j+1}, \quad (0 \quad i \neq j \quad r - 1),$$

$$\left(IF^{-1} \partial F/\partial z\right)_{i,j} = <A_z^{L_r, L_r}(F_j), \overline{F}_i> = -<\overline{F}_j, A_z^{L_0, L_r}(F_i)>$$

$$= -a_j \exp(w_j - w_0) \delta_{i,0} (\text{mod } k^C), \quad (0 \quad i \quad r - 1, r \quad j \quad n),$$

$$\left(IF^{-1} \partial F/\partial z\right)_{i,j} = <A_z^{L_j, L_{j+1}}(F_j), \overline{F}_i>$$

$$= a_i \exp(w_i - w_j) \delta_{j,r-1} (\text{mod } k^C), \quad (r \quad i \quad n, 0 \quad j \quad r - 1),$$

$$\left(IF^{-1} \partial F/\partial z\right)_{i,j} = 0 \quad (\text{mod } k^C) \quad \text{for} \quad r \quad i \neq j \quad n.$$

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Therefore, we have

\[
\alpha_m'(\partial/\partial z) = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & -5 \\
w_{1,0} & 0 & \cdots & 0 & 0 & 0_{1,n-r+1} \\
0 & w_{2,1} & \cdots & 0 & 0 & 0_{1,n-r+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & w_{r-1,r-2} & 0 & 0_{1,n-r+1} \\
0_{n-r+1,r-1} & 0 & \cdots & 0 & a_{n-r+1,n-r+1}
\end{pmatrix}
\] (3.2)

References


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