Einstein Parallel Kaehler Submanifolds in a Complex Projective Space

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Introduction

Submanifolds with parallel second fundamental form (which are simply called parallel submanifolds) have been studied by many differential geometers. In particular, parallel Kaehler submanifolds in a complex projective space are completely determined (see [1]).

In this paper, we give some characterizations of Einstein parallel Kaehler submanifolds in a complex projective space.

Let $X: M \rightarrow E^N$ be an isometric immersion of an $n$-dimensional compact Riemannian manifold into an $N$-dimensional Euclidean space. We denote by $\Delta$ and $\text{Spec}(M)=\{0<\lambda_1<\lambda_2<\cdots\}$, the Laplacian acting on differentiable functions of $M$ and the spectrum of $\Delta$, respectively. Then, $X$ can be decomposed as $X=X_0+\sum_{k \in \mathbb{N}} X_k$, where $X_k$ is a $k$-th eigenfunction of $\Delta$ of $M$, $X_0$ is a constant mapping, and the addition is convergent componentwise for the $L^2$-topology on $C^\infty(M)$. We say that the immersion is of order $\{l\}$ (or mono-order) if $X=X_0+X_l$, $l \in \mathbb{N}$, $X_l \neq 0$, and of order $\{k, l\}$ (or bi-order) if $X=X_0+X_k+X_l$, $k, l \in \mathbb{N}$, $l>k$, $X_k, X_l \neq 0$, $\cdots$ (see [4]).

Let $F: CP^m \rightarrow E^N$ be the first standard imbedding of an $m$-dimensional complex projective space of constant holomorphic sectional curvature 1 into an $N$-dimensional Euclidean space, and $i: M^n \rightarrow CP^m$ be a Kaehler immersion of an $n$-dimensional compact Kaehler manifold. We consider $\phi=F \circ i: M^n \rightarrow E^N$. Then, $\phi$ is mono-order if and only if $M$ is totally geodesic (See [3]), and totally geodesic Kaehler submanifolds are of order 1. Let $A$ be the shape operator of the immersion $i$, and define the tensor $T$ by

$$T(\xi, \eta)=\text{tr} A_\xi A_\eta$$

for $\xi, \eta \in NM$, where $NM$ is the normal bundle of $M$. Then $T$ is a symmetric bilinear mapping from $NM \times NM$ into $\mathbb{R}$. A. Ros [3] has proved that $M$ is bi-order

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if and only if $M$ is an Einstein Kaehler submanifold with $T=kg|_{NK	imes NX}$, where $k$ is some real number and $g$ is the Kaehler metric of $CP^m$. With reference to this fact, we have the following

**Theorem 1.** Let $M$ be an $n$-dimensional compact Kaehler submanifold fully immersed in $CP^m$ which is not totally geodesic. Then, the following conditions are mutually equivalent.

(i) $M$ is an Einstein parallel submanifold,

(ii) $M$ is of order \{1, 2\},

(iii) $M$ is bi-order,

(iv) $M$ is an Einstein submanifold with $T=kg|_{NK	imes NP}$,

(v) $M$ is an Einstein submanifold and $NM$ admits an Einstein Kaehler metric.

It is already shown in [4] that Einstein parallel Kaehler submanifolds which are not totally geodesic are of order \{1, 2\}. It is trivial that (ii) implies (iii). The equivalence between (iii) and (iv) is proved by A. Ros [3]. We will show that the condition (iv) implies (i) in §3. We will explain what the condition (v) means and prove the equivalence between (iv) and (v) in §2.

**Remark 1.** The equivalence between (ii) and (iii) implies that there exist no bi-order immersions other than immersions of order \{1, 2\}.

Throughout this paper, we use the following convention on the range of indices:

$A, B, \cdots = 1, \cdots, n, n+1, \cdots, m; \ a, b \cdots = 1, \cdots, n; \ \\
\alpha, \beta, \cdots = n+1, \cdots, m$.

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§1. Preliminaries.

In this section, we give some basic formulas for Kaehler submanifolds in $CP^m$. For details, see [1] and [2]. Let $M$ be an $n$-dimensional Kaehler submanifold immersed in $CP^m$. Let $TM^c$ be the complexification of the tangent bundle $TM$ of $M$. Then we have $TM^c=TM^++TM^-$ (orthogonal
sum), where the fibre $T_p M^\pm$ at $p \in M$ is the $\pm \sqrt{-1}$ eigenspace of the complex structure tensor on $T_p M^c$. In the same way, we have $NM^c = NM^+ + NM^-$ (orthogonal sum) for the complexification $NM^c$ of the normal bundle $NM$ of the immersion. We denote by $x \rightarrow \overline{x}$ the complex conjugation, so that $\overline{T_p M^\pm} = T_p M^\mp$ and $\overline{N_p M^\pm} = N_p M^\mp$. We choose a local field of unitary frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ on $CP^m$ in such a way that, restricted to $M$, $e_1, \cdots, e_n$ are tangent to $M$. With respect to the frame field on $CP^m$, let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^m\}$ be the field of dual frames. Then, the Kaehler metric of $CP^m$ is given by $\sum_\alpha \omega^\alpha \cdot \overline{\omega^\alpha}$ (in [1], the Kaehler metric of $CP^m$ is given by $2 \sum_\alpha \omega^\alpha \cdot \overline{\omega^\alpha}$) and the structure equations of $CP^m$ are given by

\begin{align}
(1.1) \quad & d\omega^a + \sum_b \omega^b \wedge \omega^{a-b} = 0, \quad \omega^a + \overline{\omega}^a = 0, \\
(1.2) \quad & d\omega^a_B + \sum_C \omega^a_C \wedge \omega^b_C = \tilde{\Omega}^a_B, \quad \tilde{\Omega}^a_B = \sum_{c,D} \tilde{R}^a_{BC\overline{D}} \omega^{a_c} \wedge \overline{\omega}^{D}.
\end{align}

Since $CP^m$ is a complex space form of constant holomorphic sectional curvature 1, we have

\begin{align}
(1.3) \quad & \tilde{R}^a_{BC\overline{D}} = (1/4)(\delta^a_B \delta^a_C + \delta^a_B \delta^a_D) - \sum_\alpha k^a_{BC} \overline{k}^a_{D\alpha}.
\end{align}

Restricting these forms to $M^n$, we have

\begin{align}
(1.4) \quad & \omega^a = 0,
\end{align}

and the Kaehler metric $g$ of $M^n$ is given by $g = \sum_a \omega^a \cdot \overline{\omega}^a$. Moreover we obtain

\begin{align}
(1.5) \quad & \omega^a_b = \sum_b k^a_{ab} \omega^b, \quad \overline{k}^a_{ab} = k^a_{ba}, \\
(1.6) \quad & d\omega^a + \sum_b \omega^a_b \wedge \omega^b = 0, \quad \omega^a + \overline{\omega}^a = 0, \\
(1.7) \quad & d\omega^a_b + \sum_c \omega^a_c \wedge \omega^b_c = \Omega^a_b, \quad \overline{\Omega}^a_b = \sum_{c,e} R_{bce}^a \omega^e \wedge \overline{\omega}^d, \\
(1.8) \quad & d\omega^a_b + \sum_r \omega^a_r \wedge \omega^b_r = \Omega^a_b, \quad \overline{\Omega}^a_b = \sum_{e,d} R_{bad}^e \omega^e \wedge \overline{\omega}^d, \\
\end{align}

From (1.5) and (1.7), we have the equation of Gauss

\begin{align}
(1.9) \quad & R^a_{bce} = (1/4)(\delta^a_b \delta^a_{de} + \delta^a_b \delta^a_{ed}) - \sum_\alpha k^a_{bce} \overline{k}^a_{ed},
\end{align}

and from (1.5), (1.6) and (1.8), we have

\begin{align}
(1.10) \quad & R^a_{bce} = (1/4)\delta^a_b \delta^a_{de} + \sum_\alpha k^a_{bce} \overline{k}^a_{ed}.
\end{align}

The Ricci tensor $S_{cd}$ and the scalar curvature $\tau$ of $M^n$ are given by

\begin{align}
(1.11) \quad & S_{cd} = (n+1)/2 \delta_{cd} - 2 \sum_{a,e} k^a_{cde} \overline{k}^a_{ed},
\end{align}
\( \tau = n(n+1) - 4 \sum_{a,b} k_{ab} \bar{k}_{ab} \).

Now, we define the covariant derivatives \( k_{abo}^\alpha \) and \( k_{a\bar{b}c}^\alpha \) of \( k_{ab}^\alpha \) by

\[
\sum_a k_{abo}^\alpha \omega^a + \sum_a k_{a\bar{b}c}^\alpha \bar{\omega}^a = dk_{ab}^\alpha - \sum_a k_{bo}^\alpha \omega^a_c - \sum_a k_{ac}^\alpha \omega^a_b + \sum_\beta k_{ab}^\beta \omega^\alpha_\beta.
\]

Then we have

\[
k_{abo}^\alpha = k_{bao}^\alpha = k_{aob}^\alpha, \quad k_{ab\bar{c}}^\alpha = 0.
\]

We can define inductively the covariant derivatives \( k_{a_1\cdots a_n\bar{b}}^\alpha \) of \( k_{a_1\cdots a_n}^\alpha \) for \( m \geq 3 \). It is clear that \((k_{a_1\cdots a_n}^\alpha)_{b} = k_{a_1\cdots a_n\bar{b}}^\alpha \) and \((k_{a_1\cdots a_n}^\alpha)_{\bar{b}} = k_{a_1\cdots a_n\bar{b}}^\alpha \).

We see that \( k_{a_1\cdots a_n}^\alpha \) is symmetric with respect to \( a_1, \cdots, a_n \).

The following formula is proved in [1]:

\textbf{LEMMA 1.}

\[
k_{a_1\cdots a_{n-1}\bar{b}}^\alpha = (m-2)/4 \sum_{r,s} k_{a_1\cdots a_{n-1}b}^\alpha - \sum_{r=1}^{n-2} \frac{1}{r! (m-r)!} \sum_{\sigma, \beta} k_{a_1\cdots a_{n-1}b}^\beta \bar{k}_{a_1\cdots a_{n-1}b}^\sigma
\]

for \( m \geq 3 \), where the summation on \( \sigma \) is taken over all permutations of \( (1, \cdots, m) \).

\( \S 2. \) Normal Einstein metric.

First, we state the following.

\textbf{DEFINITION.} We put \( R_{\bar{b}}^\alpha = \sum_a R_{\bar{b}a}^\alpha \). Then, we call this tensor on \( NM \) the normal Ricci tensor. If \( R_{\bar{b}}^\alpha = \lambda \delta_{\bar{b}}^\alpha \) for some real function \( \lambda \) on \( M \), we say that \( NM \) admits an Einstein Kaehler metric.

Let \( J \) be the complex structure of \( CP^m \). Since \( A_{J\xi} = JA_{\xi} \), \( A_{\xi}J = -JA_{\xi} \) for any \( \xi \in NM \) (See [2].), we have

\[
T(J\xi, J\eta) = T(\xi, \eta) \quad \text{for any} \quad \xi, \eta \in NM.
\]

Next, we extend \( T \) to the complex bilinear mapping from \( NM^c \times NM^c \) into \( C \). Then, we have

\[
T(N_pM^+, N_pM^+) = 0, \quad T(N_pM^-, N_pM^-) = 0 \quad \text{for any} \quad p \in M.
\]

Therefore, we can see that

\[
T = kg|_{NM \times NM} \quad \text{if and only if} \quad \sum_{a,b} k_{ab} \bar{k}_{ab} = (k/2) \delta_{ab},
\]

and \( k \) is given by \( (n(n+1) - \tau)/2(m-n) \). On the other hand, it follows from (1.10) that
Thus, we have

**PROPOSITION 1.** Let $M$ be a Kaehler submanifold in $CP^m$. Then, the following conditions are mutually equivalent.

(i) $T=kg|_{NM\times NM}$,

(ii) $NM$ admits an Einstein Kaehler metric.

§ 3. **Proof of Theorem 1.**

To complete the proof of Theorem 1, it is enough to prove the following

**PROPOSITION 2.** Let $M$ be an $n$-dimensional Einstein Kaehler submanifold in $CP^m$.

If $M$ is a submanifold with $T=kg|_{NM\times NM}$, then $M$ is a parallel submanifold.

**PROOF.** Since $M$ is Einstein, we have $\sum k_{a\overline{b}d}^\alpha \overline{k}_{a\overline{d}}^\beta = \{\|k\|^2/4n\} \delta_{d\alpha} \delta_{d\overline{\beta}}$ by (1.11) and it follows that

$$(3.1) \quad \sum k_{a\overline{b}d}^\alpha \overline{k}_{a\overline{d}}^\alpha = 0 \quad \text{(see (1.13))}.$$  

Using Lemma 1, we obtain

$$k_{a\overline{b}d}^\alpha = (1/4)\{k_{a\overline{b}}^\alpha \delta_{d\alpha} + k_{a\overline{e}}^\alpha \delta_{d\overline{\beta}} + k_{a\overline{d}}^\alpha \delta_{d\alpha}\}$$

$$- \sum_{\beta, \ell} \{k_{e\overline{a}}^\alpha k_{b\overline{c}}^\beta + k_{e\overline{d}}^\alpha k_{c\overline{a}}^\beta + k_{e\overline{c}}^\alpha k_{b\overline{d}}^\beta\} \overline{k}_{e\overline{d}}^\beta.$$  

This, together with (1.13) and (3.1), implies

$$(3.2) \quad \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha = 0.$$  

Then, we have

$$(3.3) \quad 0 = \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha + \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha$$

$$= \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha + (3/4) \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha$$

$$- 3 \sum k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha,$$

where we have used (1.13) and (3.1). If we put $H=(k_{a\overline{b}}^\alpha)$, $\tilde{T}=((\tilde{T}_{a\overline{b}}) = (\sum_k k_{a\overline{b}c}^\alpha \overline{k}_{a\overline{b}c}^\alpha)$ and denote by $\nabla$, $\tilde{\nabla}$ and $\nabla^\perp$, the $(1, 0)$-type covariant derivative, the $(0, 1)$-type covariant derivative, and the normal connection of $M$, respectively, then (3.3) implies

$$(3.4) \quad 3 ||\nabla^\perp \tilde{T}||^2 = ||\nabla^\perp H||^2 + (3/4) ||\nabla H||^2.$$

$R_{\beta}^\alpha = (n/4) \delta_{\beta}^\alpha + \sum_{a,b} k_{ab}^\alpha \overline{k}_{ab}^\alpha.$
Since \( \tilde{T}_{a\overline{b}}=(k/2)\delta_{a\beta} \), the left hand side of (3.4) vanishes, so does the right hand side. Since each term of the right hand side of (3.4) is non-negative, we obtain
\[
\nabla^\perp H=0,
\]
that is, the second fundamental form is parallel. Q.E.D.

References


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