Spectral geometry of Kaehler submanifolds of a complex projective space

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§ 0. Introduction.

Let \( X : M \to E^N \) be an isometric immersion of a compact Riemannian manifold into an \( N \)-dimensional Euclidean space. Then \( X \) can be decomposed as \( X = \sum_{k \in \mathbb{N}} X_k \), where \( X_k \) is the \( k \)-th eigenfunction of the Laplacian of \( M \) (for details, see §2). We say that the immersion is of order \( \{k_1, k_2, k_3\} \) (resp. \( \{k_1, k_2\} \) and \( k_3 \)) if \( X = X_0 + X_{k_1} + X_{k_2} + X_{k_3} \) (resp. \( X = X_0 + X_{k_1} + X_{k_2} \) and \( X = X_0 + X_{k_3} \)), where \( X_0 \) is a constant mapping and \( X_{k_1}, X_{k_2}, X_{k_3} \neq 0 \) and \( 0 < k_1 < k_2 < k_3 \).

Let \( F : \mathbb{C}P^m \to E^N \) be the standard isometric imbedding of a complex projective space into an \( N \)-dimensional Euclidean space (for details, see §1), and let \( A : M \to \mathbb{C}P^m \) be an isometric immersion of a compact Kaehler manifold into an \( m \)-dimensional complex projective space. Then \( A \) is said to be of order \( \{k_1, k_2, k_3\} \) (resp. \( \{k_1, k_3\} \) and \( k_2 \)) if the immersion \( F \circ A \) is of order \( \{k_1, k_2, k_3\} \) (resp. \( \{k_1, k_3\} \) and \( k_2 \)). A totally geodesic Kaehler submanifold of \( \mathbb{C}P^m \) is of order 1. Moreover there does not exist any compact Kaehler submanifold of order \( k_1 \geq 2 \) (see, [8], [9]), and a compact Kaehler submanifold is of order 1 if and only if it is totally geodesic. A. Ros ([2]) proved that Einstein Kaehler submanifolds with parallel second fundamental form except \( E_6/Spin(10) \times T \) in a complex projective space are of order \( \{1, 2\} \), and he characterized them by their spectra in the class of compact Kaehler submanifolds in a complex projective space. In §4, we calculate the eigenvalues of the Laplacians of \( E_6/Spin(10) \times T \) and \( E_7/E_6 \times T \). Consequently, we see that \( E_6/Spin(10) \times T \) is of order \( \{1, 2\} \), and we can say that a compact Kaehler submanifold different from a totally geodesic Kaehler submanifold in a complex projective space is of order \( \{1, 2\} \) if it is Einstein and has parallel second fundamental form (Proposition 3). Moreover we can characterize \( E_6/Spin(10) \times T \) by its spectrum in the class of compact Kaehler submanifolds in a complex projective space (Proposition 4).

Next, by applying Ros' method, we prove that \( \mathbb{C}P^n(1/3) \) and compact irreducible Hermitian symmetric spaces of rank 3 in \( \mathbb{C}P^{n+p}(1) \) are all of order \( \{1, 2, 3\} \) (Proposition 5), where \( \mathbb{C}P^n(c) \) denotes an \( m \)-dimensional complex projective space.
of holomorphic sectional curvature $c$.

The main result of this paper is the following.

**Theorem.** Let $M$ be an $n$-dimensional compact Einstein Kaehler submanifold immersed in $CP^{n+p}(1)$, and let $\tilde{M}$ be one of the Hermitian symmetric submanifolds given in Tables 2 and 3 (i.e., compact Einstein Hermitian symmetric submanifolds of degree 3).

If $\text{Spec}(M) = \text{Spec}(\tilde{M})$, then $M$ is congruent to $\tilde{M}$.

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§1. Preliminaries.

Let $\text{HM}(m+1) = \{A \in gl(m+1, C) \mid \bar{A} = {}^tA\}$ be the space of $(m+1)\times(m+1)$-Hermitian matrices. We define on $\text{HM}(m+1)$ an inner product $g$ by

$$g(A, B) = 2\text{tr}AB$$

for $A, B \in \text{HM}(m+1)$.

We consider the submanifold $CP^m = \{A \in \text{HM}(m+1) \mid AA = A, \text{tr}A = 1\}$. It is known that $CP^m$, with the metric induced from $g$ on $\text{HM}(m+1)$, is isometric to the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. The tangent space and the normal space at any point $A$ of $CP^m$ are given respectively by

$$T_A(CP^m) = \{X \in \text{HM}(m+1) \midXA+AX=X\},$$

$$T_A^\perp(CP^m) = \{Z \in \text{HM}(m+1) \mid ZA = AZ\}.$$

Let $D, \tilde{\nabla}, \tilde{\sigma}, \tilde{\nabla}^\perp, \tilde{\Lambda}, \tilde{H}$ be the Riemannian connection of $\text{HM}(m+1)$, the induced connection in $CP^m$, the second fundamental form of the immersion, the normal connection, the Weingarten endomorphism, the mean curvature vector of $CP^m$ in $\text{HM}(m+1)$, respectively.

A. Ros [8, 9] obtained the following facts.

(1.1) $\tilde{\sigma}(X, Y) = (XY+YX)(I-2A),$

(1.2) $\tilde{\Lambda}_ZX = (XZ-ZX)(I-2A),$

(1.3) $\tilde{H} = \frac{1}{2m}[I-(m+1)A],$

(1.4) $JX = \sqrt{-1}(I-2A)X,$

(1.5) $\tilde{\sigma}(JX, JY) = \tilde{\sigma}(X, Y), \quad \tilde{\nabla}\tilde{\sigma} = 0,$

(1.6) $g(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W)) = \frac{1}{2}g(X, Y)g(V, W)+\frac{1}{4}(g(X, W)g(Y, V)$.
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\[ +g(X, V)g(Y, W) + g(X, JW)g(Y, JV) \]
\[ + g(X, JV)g(Y, JW) \]
\[
\tilde{\Lambda}_{\overline{\sigma}(X, Y)}V = \frac{1}{2}g(X, Y)V + \frac{1}{4}\{g(Y, V)X + g(X, V)Y + g(JY, V)JX + g(JX, V)JY\} \]

(1.7) \[ g(\tilde{\sigma}(X, Y), I) = 0, \quad g(\tilde{\sigma}(X, Y), A) = -g(X, Y), \]

where \( I \) is the \((m+1)\times(m+1)\)-identity matrix, \( J \) is the complex structure of \( CP^m \), \( X, Y, V, W \in T_A(CP^m) \) and \( Z \in T^\perp_A(CP^m) \).

§ 2. The order of an immersion.

Let \( X: M^n \rightarrow E^N \) be an isometric immersion of an \( n \)-dimensional compact Riemannian manifold into the \( N \)-dimensional Euclidean space. Let \( \Delta \) be the Laplacian of \( M \) acting on differentiable functions and \( \text{Spec}(M) = \{0 < \lambda_1 = \cdots = \lambda_s < \lambda_{s+1} = \cdots\} \) be the spectrum of \( \Delta \). Then we have the orthogonal decomposition \( X = \sum_k X_k, k \in \mathbb{N} \), where \( X_k : M \rightarrow E^N \) is a differentiable mapping satisfying \( \Delta X_k = \lambda_k X_k \), and the addition is convergent, componentwise, for the \( L^2 \)-topology on \( C^\infty(M) \).

We have the relations

(2.1) \[ \Delta X = -nH = \sum_{k \geq 1} \lambda_k X_k, \]
(2.2) \[ \Delta^2 X = -n\Delta H = \sum_{k \geq 1} \lambda_k^2 X_k, \]
(2.3) \[ \Delta^3 X = -n\Delta^2 H = \sum_{k \geq 1} \lambda_k^3 X_k, \]

where \( H \) is the mean curvature vector of \( M \) in \( E^N \).

Let \( k_1, k_2, k_3 \in \mathbb{N} \) with \( 0 < k_1 < k_2 < k_3 \). We say that the immersion \( X \) is of order \( k_1 \) (resp. \( k_1, k_2 \)) and \( \{k_1, k_2, k_3\} \) if \( X = X_0 + X_{k_1} \) (resp. \( X_0 + X_{k_1} + X_{k_2} \) and \( X_0 + X_{k_1} + X_{k_2} + X_{k_3} \)) and \( X_{k_1}, X_{k_2}, X_{k_3} \neq 0 \).

§ 3. Kaehler submanifolds.

Let \( M^n \) be an \( n \)-dimensional compact Kaehler submanifold immersed in the \((n+p)\)-dimensional complex projective space \( CP^{n+p} \), and let \( A : M^n \rightarrow CP^{n+p} \) be the immersion. Let \( E_1, \ldots, E_n, E_1 = JE_1, \ldots, E_n = JE_n, \xi_1, \ldots, \xi_p, \xi_1 = J\xi_1, \ldots, \xi_p = J\xi_p \) be a local field of orthonormal frames of \( CP^{n+p} \), such that, restricted to \( M \), \( E_1, \ldots, E_n, E_1, \ldots, E_n \) are tangent to \( M \). Let \( \nabla, \sigma, \nabla^\perp \) and \( A \) be the Riemannian connection, the second fundamental form, the normal connection
and the Weingarten endomorphism of $M$ in $CP^{n+p}$ respectively, and $H$ the mean curvature vector of $M$ in $HM(n+p+1)$.

Throughout this paper, we use the following convention on the range of indices: $i, j, k, l, \cdots = 1, \cdots, n, 1^*, \cdots, n^*, \lambda, \mu, \cdots = 1, \cdots, p, 1^*, \cdots, p^*, A, B, C, \cdots = 1, \cdots, n, n+1, \cdots, n+p, a, b, c, \cdots = 1, \cdots, n, \alpha, \beta, \gamma, \cdots = 1, \cdots, p$. Then, the immersion $X$ is of order $k_1$ if and only if $M^n$ is totally geodesic and the immersion $X$ is of order $\{k_1, k_2\}$ if and only if

\begin{equation}
\Delta H = (\lambda_{k_1} + \lambda_{k_2})H + \frac{\lambda_{k_1}\lambda_{k_2}}{2n}(X - X_0)
\end{equation}

(see [9], and in the same way as in p. 440 of [9]) we can see that the immersion $X$ is of order $\{k_1, k_2, k_3\}$ if and only if

\begin{equation}
\Delta^2 H = (\lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3})\Delta H - (\lambda_{k_1}\lambda_{k_2} + \lambda_{k_2}\lambda_{k_3} + \lambda_{k_3}\lambda_{k_1})H - \frac{\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}{2n}(X - X_0).
\end{equation}

We prepare the following Lemma.

**Lemma 1** (A. Ros [9]).

\begin{equation}
H = \frac{1}{2n} \sum \bar{\theta}(E_i, E_i),
\end{equation}

\begin{equation}
\Delta H = (n+1)H + \frac{1}{n} \sum \bar{\theta}(\Lambda_{\sigma(E_i, E_j)}E_i, E_j)
\end{equation}

\begin{equation}
- \frac{1}{n} \sum \bar{\theta}(\sigma(E_i, E_j), \sigma(E_i, E_j)).
\end{equation}

This is obtained by using (1.7) and the fact that $M$ is minimal in $CP^{n+p}$ and that $CP^{n+p}$ has parallel second fundamental form.

The normal space of $M$ in $CP^{n+p}$ at $x$ is denoted by $T^\perp_x(M)$. We define the tensor $T: T^\perp_x \times T^\perp_x \rightarrow R$ by

\begin{equation}
T(\xi, \eta) = tr A_\xi A_\eta \quad \text{for all } \xi, \eta \in T^\perp_x(M).
\end{equation}

Then, A. Ros [9] obtained the following result.

**Proposition 1.** Let $M$ be an $n$-dimensional compact Kaehler submanifold in $CP^{n+p}$ such that the immersion $A: M \rightarrow CP^{n+p}$ is full. Then $M$ is a submanifold of order $\{k_1, k_2\}$ in $HM(n+p+1)$ if and only if $M$ is an Einstein submanifold with $T = kg|_{T^\perp \times T^\perp}$ for some real number $k$.

If the immersion is full, the constant part $X_0$ of $X$ is given by $X_0 = (1/(n+p+1))I$ (see [9]), where $I$ is the $(n+p+1) \times (n+p+1)$-identity matrix.


Let $(G, K)$ be a Riemannian symmetric pair. Let $g$ and $\mathfrak{f}$ be the Lie alge-
bras of $G$ and $K$, respectively. Then we have the canonical decomposition $g = t + m$. Let $a$ be a Cartan subalgebra of $(G, K)$, i.e., a maximal Abelian subalgebra of $g$ contained in $m$, and let $t$ be a maximal Abelian subalgebra of $g$ containing $a$. Then we have the direct sum decomposition $t = a + b$. We define the involution $S$ by

$$S(H_1 + H_2) = -H_1 + H_2, \quad H_1 \in b, \quad H_2 \in a,$$

and define $\overline{H}$ by

$$\overline{H} = \frac{1}{2}(H + S(H)), \quad H \in t.$$

Let $\Sigma(G)$ be the set of all roots of $G$ with respect to $t$, and define $\Sigma_0(G), \Sigma(G, K), \Sigma^+(G, K)$ by

$$\Sigma_0(G) = \Sigma(G) \cap b, \quad \Sigma(G, K) = \{ \alpha \in \Sigma(G) - \Sigma_0(G) \},$$

$$\Sigma^+(G, K) = \{ \gamma \in \Sigma(G, K) ; \gamma > 0 \},$$

respectively. Next, we define $\Gamma(G), Z(G), D(G), \Gamma(G, K), Z(G, K), D(G, K)$ by

$$\Gamma(G) = \{ H \in t ; \exp H = e \in T \},$$

$$Z(G) = \{ \lambda \in t ; (\lambda, H) \in Z \text{ for all } H \in \Gamma(G) \},$$

$$D(G) = \{ \lambda \in Z(G) ; (\lambda, H) \geq 0 \},$$

$$\Gamma(G, K) = \{ H \in a ; \exp H \in K \},$$

$$Z(G, K) = \{ \lambda \in a ; (\lambda, H) \in Z \text{ for all } H \in \Gamma(G, K) \},$$

$$D(G, K) = \{ \lambda \in Z(G, K) ; (\lambda, \gamma) \geq 0 \text{ for all } \gamma \in \Sigma^+(G, K) \},$$

respectively, where $T$ is the maximal torus generated by $t$, and $e$ is the identity, and $(, )$ is the inner product on $t$.

Let $\Pi(G) = \{ \alpha_1, \cdots, \alpha_l \}$ be the fundamental root system, and let $N_1, \cdots, N_l$ be the fundamental weights of $g$ defined by

$$\frac{2(N_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad \text{for } N_i \in t,$$

where $l = \text{rank}(G)$. Let $M_i$ be the fundamental weights of $(g, t)$ defined by

$$M_i = \begin{cases} 2N_i, & \text{if } p\alpha_i = \alpha_i, \quad (\alpha_i, \Pi_0(G)) = \{0\} \\ N_i, & \text{if } p\alpha_i = \alpha_i, \quad (\alpha_i, \Pi_0(G)) \neq \{0\} \\ N_i + N_j, & \text{if } p\alpha_i = \alpha_j, \quad \alpha_i \neq \alpha_j, \end{cases}$$

where $\Pi_0(G) = \Pi(G) \cap \Sigma_0(G)$ and $p$ is the Satake involution. We put $\delta(G) = \sum_i N_i$.

We review the following facts (see [12]).

**FACT 1.** Let $(G, K)$ be a compact symmetric pair such that $G/K$ is simply-
connected. Then
\[ D(G, K) = \left\{ \sum_{i=1}^{l} m_i M_i \mid m_i \in \mathbb{Z}, m_i \geq 0 \ (1 \leq i \leq l) \right\}. \]

FACT 2. Let \( \rho \) be a spherical representation of \( G \) with respect to \( K \). Then the highest weight \( \lambda(\rho) \) of \( \rho \) with respect to \( t \) belongs to \( D(G, K) \).

FACT 3. The mapping \( \rho \mapsto \lambda(\rho) \) is bijective.

Now we can compute the eigenvalues of \( \Delta \) for \( E_6/\text{Spin}(10) \times T \) and \( E_7/E_6 \times T \).

i) \( E_6/\text{Spin}(10) \times T \): We put \( G = E_6 \) and \( K = \text{Spin}(10) \times T \). The fundamental roots are given by (see [2])
\[
\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + \cdots + e_7), \quad \alpha_2 = e_1 + e_2, \\
\alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \quad \alpha_5 = e_4 - e_3, \quad \alpha_6 = e_5 - e_4.
\]
where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^8 \) for \( i = 1, \ldots, 8 \). The fundamental weights of \( \mathfrak{g} \) are given by
\[
N_1 = \frac{2}{3}(e_6 - e_1 - e_8), \\
N_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_6 - e_7 + e_8), \\
N_3 = \frac{5}{6}(e_6 - e_1 - e_8) + \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_6), \\
N_4 = e_3 + e_4 + e_5 + e_6 - e_7 + e_8, \\
N_5 = \frac{2}{3}(e_6 - e_1 - e_8) + e_4 + e_5, \\
N_6 = \frac{1}{3}(e_6 - e_1 - e_8) + e_5, \\
\]
and
\[
\delta(G) = \sum_i N_i = e_2 + 2e_3 + 3e_4 + 4e_5 + 4(e_6 - e_7 - e_8).
\]
From diagram 1, the fundamental weights of $(\mathfrak{g}, \mathfrak{t})$ are given by

\[ M_1 = N_1 + N_2 = e_6 - e_7 - e_8 + e_8, \]
\[ M_2 = N_2 = \frac{1}{2} (e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8). \]

It follows from Facts 1, 2 and 3 that $\lambda(\rho) = m_1 M_1 + m_2 M_2$. Therefore the Freudenthal's formula implies that the eigenvalue $A_\rho$ of the Casimir operator of an irreducible representation $\rho$ is given by

\[ A_\rho = \frac{1}{2} (\lambda(\rho) + 2\delta(G), \lambda(\rho)) \]
\[ = 2m_1(m_1 + m_2 + 8) + m_2(m_2 + 11). \]

Since the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ of $\Delta$ are given by $A_\rho$'s, we see that

\[ \lambda_1 = 12 \quad (m_1 = 0, m_2 = 1), \]
\[ \lambda_2 = 18 \quad (m_1 = 1, m_2 = 0). \]

ii) $E_7/E_6 \times T$: We put $G = E_7$ and $K = E_6 \times T$. The fundamental roots are given by

\[ \alpha_1 = \frac{1}{2} (e_1 + e_8) - \frac{1}{2} (e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \]
\[ \alpha_2 = e_1 + e_2, \quad \alpha_3 = e_2 - e_1, \quad \alpha_4 = e_3 - e_2, \]
\[ \alpha_5 = e_4 - e_3, \quad \alpha_6 = e_5 - e_4, \quad \alpha_7 = e_6 - e_5. \]

The fundamental weights of $\mathfrak{g}$ are given by

\[ N_1 = e_8 - e_7, \]
\[ N_2 = \frac{1}{2} (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 2e_7 + 2e_8), \]
\[ N_3 = \frac{1}{2} (-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3e_7 + 3e_8), \]
\[ N_4 = e_3 + e_4 + e_5 + e_6 + 2(e_8 - e_7), \]
\[ N_5 = \frac{1}{2} (2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 - 3e_6), \]
\[ N_6 = e_5 + e_6 - e_7 + e_8, \]
\[ N_7 = e_6 + \frac{1}{2} (e_8 - e_7), \]

and

\[ 2\delta(G) = 2e_2 + 4e_3 + 6e_4 + 8e_5 + 10e_6 - 17e_7 + 17e_8. \]
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Diagram 2.

From diagram 2, the fundamental weights of \((g, f)\) are given by

\[
\begin{align*}
M_1 &= N_1 = e_8 - e_7, \\
M_6 &= N_6 = e_5 + e_6 - e_7 + e_8, \\
M_7 &= 2N_7 = 2e_6 + e_8 - e_7.
\end{align*}
\]

Hence the highest weight is given by

\[
\lambda(\rho) = m_1M_1 + m_2M_6 + m_3M_7
\]

where \(m_1, m_2, m_3 \in \mathbb{Z}, \ m_1, m_2, m_3 \geq 0\). Therefore the Freudenthal's formula implies that

\[
A_\rho = \frac{1}{2}(\lambda(\rho) + 2\delta(G), \lambda(\rho)) = m_1^2 + 2m_2^2 + 3m_3^2 + 2m_1m_2 + 4m_2m_3 + 2m_3m_1 + 17m_1 + 26m_2 + 27m_3.
\]

Thus we see that the eigenvalues \(0 < \lambda_1 < \lambda_2 < \cdots\) of \(\Delta\) are given by

\[
\begin{align*}
\lambda_1 &= 18 \quad (m_1=1, \ m_2=m_3=0), \\
\lambda_2 &= 28 \quad (m_1=m_3=0, \ m_2=1), \\
\lambda_3 &= 30 \quad (m_1=m_2=0, \ m_3=1),
\end{align*}
\]

\[\cdots\]

§ 5. Spectral geometry for Kaehler submanifolds I.

First we state the following.

**Lemma 2** ([9]).

\[
\begin{align*}
(5.1) \quad & g(A, A) = 2, \\
(5.2) \quad & g(A, H) = -1, \\
(5.3) \quad & g(A, \Delta H) = -(n+1), \\
(5.4) \quad & g(H, H) = \frac{n+1}{2n}, \\
(5.5) \quad & g(H, \Delta H) = \frac{(n+1)^2}{2n} + \frac{1}{2n^2} \|\sigma\|^2, \\
(5.6) \quad & g(\Delta H, \Delta H) = \frac{(n+1)^3}{2n} + \frac{n+1}{n^2} \|\sigma\|^2 + \frac{1}{n^2} \|T\|^2 + \frac{1}{n^2} \text{tr}(\sum \lambda \Lambda_{\lambda}^2)^2,
\end{align*}
\]
where \( \Lambda_\lambda = \Lambda_{\xi_\lambda} \).

Note that \( \int_M g(X_r, X_s) = 0 \) for \( r \neq s \), and put \( a_k = \int_M g(X_k, X_k) \). Then from (2.1) and (2.2) we have

\[-2n \int_M g(X, H) = \sum_{k \geq 1} \lambda_k a_k,\]
\[4n^2 \int_M g(H, H) = \sum_{k \geq 1} \lambda_k^2 a_k,\]
\[4n^2 \int_M g(H, \Delta H) = \sum_{k \geq 1} \lambda_k^3 a_k,\]
\[4n^2 \int_M g(\Delta H, \Delta H) = \sum_{k \geq 1} \lambda_k^4 a_k.\]

We put

\[\Phi_1 = 4n^2 \int_M g(H, H) + 2n \lambda_1 \int_M g(X, H),\]
\[\Phi_2 = 4n^2 \int_M g(H, \Delta H) - 4n^2 \lambda_1 \int_M g(H, H),\]
\[\Phi_3 = 4n^2 \int_M g(\Delta H, \Delta H) - 4n^2 \lambda_1 \int_M g(H, \Delta H),\]
\[\Phi_4 = 4n^2 \int_M g(\Delta H, \Delta H) - 4n^2 (\lambda_1 + \lambda_2) \int_M g(H, \Delta H) + 4n^2 \lambda_1 \lambda_2 \int_M g(H, H),\]
\[\Phi_5 = 4n^2 \int_M g(\Delta H, \Delta H) - 4n^2 (\lambda_1 + \lambda_2 + \lambda_3) \int_M g(H, \Delta H) + 4n^2 (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \int_M g(H, H) + 2n \lambda_1 \lambda_2 \lambda_3 \int_M g(X, H).\]

Then we get

(5.7) \( \Phi_1 = \sum_{k \geq 2} \lambda_k (\lambda_k - \lambda_1) a_k \geq 0, \)
(5.8) \( \Phi_2 = \sum_{k \geq 2} \lambda_k^2 (\lambda_k - \lambda_1) a_k \geq 0, \)
(5.9) \( \Phi_3 = \sum_{k \geq 2} \lambda_k^3 (\lambda_k - \lambda_1) a_k \geq 0, \)
(5.10) \( \Phi_4 = \Phi_3 - \lambda_2 \Phi_2 = \sum_{k \geq 2} \lambda_k^2 (\lambda_k - \lambda_1)(\lambda_k - \lambda_2) a_k \geq 0, \)
(5.11) \( \Phi_5 = \Phi_4 - \lambda_3 (\Phi_2 - \lambda_2 \Phi_1) = \sum_{k \geq 3} \lambda_k (\lambda_k - \lambda_1)(\lambda_k - \lambda_2)(\lambda_k - \lambda_3) a_k \geq 0. \)

We put

(5.12) \( \Phi_6 = \Phi_2 - \lambda_2 \Phi_1 = \sum_{k \geq 2} \lambda_k (\lambda_k - \lambda_2)(\lambda_k - \lambda_1) a_k \geq 0. \)

The equality in (5.7) holds if and only if the immersion is of order 1, the
equality in [5.12] holds if and only if the immersion is of order 1 or \{1, 2\}, and the equality in [5.11] holds if and only if the immersion is of order 1 or \{1, 2\} or \{1, 3\} or \{2, 3\} or \{1, 2, 3\}.

Thus we have

**Proposition 2** (N. Ejiri, A. Ros, see [9]). Let \( M \) be an \( n \)-dimensional compact Kaehler submanifold immersed in \( CP^m \). Then

\[
\lambda_1 \leq n+1.
\]

The equality holds if and only if \( M \) is totally geodesic (that is, of order 1).

**Proposition 3.** Let \( M \) be an \( n \)-dimensional compact Kaehler submanifold immersed in \( CP^m \).

If \( \lambda_1 = \frac{\tau}{\langle \text{vol}(M) \rangle} \) and \( M \) is not totally geodesic, then

\[
\lambda_2 \leq n+2.
\]

The equality holds if and only if \( M \) is Einstein and the second fundamental form of the immersion is parallel (that is, of order \{1, 2\}).

**Proof.** In Corollary 5.4 in [9], under the same assumptions as Proposition 3, it is proved that \( \lambda_2 \leq n+2 \) and the equality holds only if \( M \) is Einstein and the second fundamental form of the immersion is parallel. Hence it is enough to prove that if \( M \) is an Einstein parallel submanifold, then \( \lambda_2 = n+2 \). But, from Theorem 7.4 in [6], all Einstein parallel submanifolds are listed in Table 1, which, together with the result obtained in \S 4, shows that \( \lambda_2 = n+2 \). Using Lemma 2 and (5.12) we see that \( \lambda_2 = n+2 \) if and only if the equality in (5.12) holds since \( M \) is not totally geodesic. But since the equality in (5.12) holds if and only if \( M \) is of order \{1, 2\}, the proof of Proposition 3 is accomplished.

### Table 1. Einstein Kaehler submanifolds of degree 2.

<table>
<thead>
<tr>
<th>Submanifold</th>
<th>dim(_c)</th>
<th>( p )</th>
<th>( \tau )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 = CP^n(1/2) )</td>
<td>( n )</td>
<td>( n(n+1)/2 )</td>
<td>( n(n+1)/2 )</td>
<td>( (n+1)/2 )</td>
<td>( n+2 )</td>
</tr>
<tr>
<td>( M_2 = Q^n )</td>
<td>( n )</td>
<td>( 1 )</td>
<td>( n^2 )</td>
<td>( n )</td>
<td>( n+2 )</td>
</tr>
<tr>
<td>( M_3 = CP^n \times CP^n )</td>
<td>( 2n )</td>
<td>( n^2 )</td>
<td>( 2n(n+1) )</td>
<td>( n+1 )</td>
<td>( 2n+2 )</td>
</tr>
<tr>
<td>( M_4 = U(s+2)/U(2) \times U(s) )</td>
<td>( s \geq 3 )</td>
<td>( 2s )</td>
<td>( s(s+1)/2 )</td>
<td>( s+2 )</td>
<td>( 2s+2 )</td>
</tr>
<tr>
<td>( M_5 = SO(10)/U(5) )</td>
<td>10</td>
<td>5</td>
<td>80</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>( M_6 = E_6/Spin(10) \times T )</td>
<td>16</td>
<td>10</td>
<td>192</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>
Since dimension, $\text{vol}(M)$, and $\int_M \tau$ are spectral invariants, from Proposition 3 and Table 1, we have

PROPOSITION 4. Let $M$ be an $n$-dimensional compact Kaehler submanifold immersed in $CP^m$. If $\text{Spec}(M)=\text{Spec}(M_i)$ for some $i=1, \cdots, 6$, then $M$ is congruent to the standard imbedding of $M_i$, where $M_i$ is one of the Hermitian symmetric spaces given in Table 1.

REMARK. Proposition 4 for $i=1, \cdots, 5$ is obtained in [9].

The following formulas are well-known (for example, see [7], [9]),

(5.13) $\tau = n(n+1)-\|\sigma\|^2$,

(5.14) $\|S\|^2 = \frac{1}{2}n(n+1)-\|\sigma\|^2+\text{tr}(\sum_\lambda \Lambda_\lambda^2)^2$,

(5.15) $\|R\|^2 = 2n(n+1)-4\|\sigma\|^2+2\|T\|^2$,

(5.16) $-\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla\sigma\|^2 + \frac{n^2+2}{2}\|\sigma\|^2 - 2\text{tr}(\sum_\lambda \Lambda_\lambda^2)^2 - \|T\|^2$,

(5.17) $\frac{n(n+1)}{2}\|R\|^2 \geq 2n\|S\|^2 \geq \tau^2$.

The first equality in (5.17) holds if and only if $M$ has constant holomorphic sectional curvature, and the second equality in (5.17) holds if and only if $M$ is Einstein.

From (5.13), (5.14) and (5.17), we have

(5.18) $\text{tr}(\sum_\lambda \Lambda_\lambda^2)^2 \geq \frac{1}{2n}\|\sigma\|^4$.

The equality holds if and only if $M$ is Einstein.

LEMMA 3. Let $M$ be an $n$-dimensional compact Kaehler submanifold immersed in $CP^m$ with the following properties:

i) $\lambda_1 = \frac{\int_M \tau}{n\text{vol}(M)}$,

ii) $\lambda_2 = \frac{(n+3)\lambda_1 - \int_M (\|R\|^2+2\|S\|^2)/(n\text{vol}(M))}{n+1-\lambda_1} + \lambda_1$,

iii) $\nabla\sigma \neq 0$.

Then

$\lambda_3 \leq n+3$.

The equality holds only if the immersion is of order $\{1, 3\}$ or $\{2, 3\}$ or $\{1, 2, 3\}$. Moreover, $\lambda_1, \lambda_2, \lambda_3$ and $\|\sigma\|^2$ are given as follows: For the case of order $\{1, 3\}$,
\[
\lambda_1 = \frac{n(n+p+1)-p}{n+2p}, \quad \lambda_2 = n+1, \quad \lambda_3 = n+3.
\]

(5.19)

\[
\|\sigma\|^2 = \frac{np(n+3)}{n+2p},
\]

and, for the case of order \{2, 3\},

\[
\lambda_1 = \frac{2n(n+1)+p(n-3)}{2n+3p}, \quad \lambda_2 = \frac{2n(n+p+1)}{2n+3p}, \quad \lambda_3 = n+3.
\]

(5.20)

\[
\|\sigma\|^2 = \frac{2np(n+3)}{2n+3p},
\]

where \(p\) is the full codimension.

**PROOF.** Using Lemma 2, (5.13), (5.14) and (5.15), we have

\[
\Phi_5 = 2n \text{vol}(M)((n+1)(n+2)(n+3)-(n+1)(n+2)(\lambda_1+\lambda_2+\lambda_3)
\]

\[
+(n+1)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)-\lambda_1\lambda_2\lambda_3)
\]

\[
+2(\lambda_1+\lambda_2+\lambda_3-4n-8)\int_M \tau + 2\int_M (\|R\|^2 + 2\|S\|^2).
\]

From the assumptions i) and ii), we get

\[
\Phi_5 = 2n \text{vol}(M)(n+1-\lambda_1)(n+2-\lambda_2)(n+3-\lambda_3).
\]

From the assumption iii), Proposition 2, Proposition 3 and (5.11), we have

\[
\lambda_3 \leq n+3.
\]

If the immersion is of order \(\{k_1, k_2\}\), then the following holds (see [9]):

\[
\lambda_{k_1} + \lambda_{k_2} = n+1 + \frac{(n+p)\|\sigma\|^2}{np}.
\]

(5.21)

It follows from Proposition 1 that \(M\) is an Einstein Kaehler submanifold with \(T=kg\). Hence we obtain

\[
\lambda_1 = \frac{\tau}{n} = \frac{n(n+1)-\|\sigma\|^2}{n}.
\]

(5.22)

Since \(\|T\|^2 = \|\sigma\|^4/2p\) (see [9]), from (5.14), (5.15) and (5.18), we have

\[
\int_M (\|R\|^2 + 2\|S\|^2)
\]

\[
\text{vol}(M) = (n+1)(n+3) - \frac{2(n+3)\|\sigma\|^2}{n} + \frac{(n+p)\|\sigma\|^4}{n^2p}.
\]

(5.23)

From (5.21), (5.22), (5.23) and \(\lambda_3 = n+3\), we have (5.19) for the case of order \{1, 3\}, and (5.20) for the case of order \{2, 3\}. Q.E.D.
§6. Spectral geometry for Kaehler submanifolds II.

In this section, we investigate the order of $CP^{n}(1/3)$ and compact irreducible Hermitian symmetric spaces of rank 3. We choose a local field of unitary frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ on $CP^{n+p}$ in such a way that, restricted to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. With respect to the frame field on $CP^{n+p}$, let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+p}\}$ be the field of dual frames. Then the Kaehler metric of $CP^{n+p}$ is given by $\Sigma_{A=1}^{n+p} \omega^A \cdot \overline{\omega}^A$ and the structure equations of $CP^{n+p}$ are given by

\begin{align}
(6.1) \quad & d\omega^A + \sum_B \omega_B^A \wedge \omega^B = 0, \quad \omega^A + \overline{\omega}^A = 0, \\
(6.2) \quad & d\omega^A + \sum_B \omega_B^A \wedge \omega^B = \tilde{\Omega}_B^A, \quad \tilde{\Omega}_B^A = \sum_{C,D} R_{BC\overline{D}}^{A} \omega^C \wedge \overline{\omega}^D.
\end{align}

Since $CP^{n+p}$ is a complex space form of constant holomorphic sectional curvature 1, we have

\begin{equation}
R_{BC\overline{D}}^{A} = \frac{1}{4} (\delta_{B}^{A} \delta_{CD} + \delta_{C}^{A} \delta_{BD}).
\end{equation}

Restricting these forms to $M^n$, we have

\begin{equation}
(6.3) \quad \omega^a = 0,
\end{equation}

and the Kaehler metric $g$ of $M^n$ is given by $g = \sum_a \omega^a \cdot \overline{\omega}^a$. Moreover we obtain

\begin{align}
(6.5) \quad & \Omega_B^A = \sum_{c,d} R_{BC\overline{D}}^{A} \omega^C \wedge \overline{\omega}^D, \\
(6.6) \quad & d\omega^a + \sum_b \omega_b^a \wedge \omega^b = 0, \quad \omega^a + \overline{\omega}^a = 0, \\
(6.7) \quad & d\omega^a + \sum_c \omega_c^a \wedge \omega^c = \Omega_B^a, \quad \Omega_B^a = \sum_{c,d} R_{BC\overline{D}}^{a} \omega^C \wedge \overline{\omega}^D, \\
(6.8) \quad & d\omega^a + \sum_{c,d} \omega_c^a \wedge \omega^d = \Omega_B^a, \quad \Omega_B^a = \sum_{c,d} R_{BC\overline{D}}^{a} \omega^C \wedge \overline{\omega}^D.
\end{align}

From (6.5) and (6.7), we have the equation of Gauss

\begin{equation}
(6.9) \quad R_{Bcd}^a = \frac{1}{4} (\delta_b^a \delta_{cd} + \delta_{cd} \delta_{ba}) - \sum_{c,d} k_{ac}^b \overline{k}_{ad}^a,
\end{equation}

and from (6.5) (6.6) and (6.8), we have

\begin{equation}
(6.10) \quad R_{c\overline{d}}^a = \frac{1}{4} \delta_{c\overline{d}}^a \delta_{ba} + \sum_{a,c,d} k_{ac}^b \overline{k}_{ad}^a.
\end{equation}

The Ricci tensor $S_{c\overline{d}}$ and the scalar curvature $\tau$ of $M^n$ are given by

\begin{align}
(6.11) \quad & S_{c\overline{d}} = \frac{n+1}{2} \delta_{c\overline{d}} - 2 \sum_{a,c,d} k_{ac}^b \overline{k}_{ad}^a, \\
(6.12) \quad & \tau = n(n+1) - 4 \sum_{a,c,d} k_{ac}^b \overline{k}_{ad}^a.
\end{align}
Now, we define the covariant derivatives $k_{abc}^\alpha$ and $k_{ab\overline{c}}^\alpha$ of $k_{ab}^\alpha$ by
\[
\sum_c k_{abc}^\alpha \omega_c + \sum_c k_{ab\overline{c}}^\alpha \overline{\omega}_c = dk_{ab}^\alpha - \sum_c k_{cb}^\alpha \omega_a^c - \sum_c k_{ac}^\alpha \omega_b^c + \sum_\beta k_{\beta ab}^\alpha \omega_\beta^\alpha.
\]

Then we have
\[ (k_{a_1\ldots a_m}^\alpha)_{b} = k_{a_1\ldots a_m b}^\alpha \quad \text{and} \quad (k_{a_1\ldots a_m}^\alpha)_{\overline{b}} = k_{a_1\ldots a_m \overline{b}}^\alpha. \]

We see that $k_{a_1\ldots a_m}^\alpha$ is symmetric with respect to $a_1, \ldots, a_m$. The following formula is proved in [6]:

**Lemma 4.**
\[
k_{a_1\ldots a_m b}^\alpha = \frac{m-2}{4} \sum_{r=1}^m k_{a_1\ldots \hat{a}_r\ldots a_m}^\alpha \delta_{a_r b} - \sum_{r=1}^{m-2} \frac{1}{r!(m-r)!} \sum_{\sigma, \beta, c} k_{ca_{\sigma(1)}\ldots a_{\sigma(\gamma)}}^\alpha k_{a_{\sigma(\tau+1)}\ldots a_{\sigma(m)}}^\beta \overline{k}_{cb}^\beta,
\]
for $m \geqslant 3$, where the summation on $\sigma$ is taken over all permutations of $(1, \ldots, m)$.

Let $T_x(M)$ be the tangent space to $M$ at $x$ and $T_x^\mathbb{C}(M)$ its complexification. Let $T_x^{\mathbb{C},0}(M) = \{X - \sqrt{-1}JX \mid X \in T_x(M)\}$ and $T_x^{\mathbb{C},1}(M) = \{X + \sqrt{-1}JX \mid X \in T_x(M)\}$. Then
\[
T_x^\mathbb{C}(M) = T_x^{\mathbb{C},0}(M) + T_x^{\mathbb{C},1}(M).
\]

The similar results hold for $CP^{n+p}$. Suppose that the relation between $e_A$ and $E_A$ is given by
\[
e_A = \frac{1}{2} (E_A - \sqrt{-1} E_A), \quad \overline{e_A} = \frac{1}{2} (E_A + \sqrt{-1} E_A).
\]

Then, the relation between $h_{ab}^\alpha$ and $k_{ab}^\alpha$ is given by (see [7])
\[
k_{ab}^\alpha = h_{ab}^\alpha - \sqrt{-1} h_{ab\overline{c}}^\overline{c}, \quad \overline{k}_{ab}^\alpha = h_{ab}^\alpha + \sqrt{-1} h_{ab\overline{c}}^\overline{c}.
\]

Moreover we can see that
\[
k_{abc}^\alpha = h_{abc}^\alpha - \sqrt{-1} h_{abc\overline{c}}^\overline{c}, \quad \overline{k}_{abc}^\alpha = h_{abc}^\alpha + \sqrt{-1} h_{abc\overline{c}}^\overline{c}.
\]

Thus we have
\[ \|\sigma\|^2 = \sum_{i,t,j} h_{ij}^t h_{ij}^t = 4 \sum_{a,b} k_a^\sigma \bar{k}_b^\sigma, \]

(6.17)

\[ \|T\|^2 = \sum_{i,p,k,l} h_{ij}^t h_{i}^p h_{kl}^t h_{kl}^t = 8 \sum_{\alpha,\beta,a,b,c,d} k_{a\beta}^\alpha \bar{k}_{b\alpha}^\beta k_{c\beta}^\beta \bar{k}_{d\alpha}^\alpha, \]

\[ \|\nabla\sigma\|^2 = \sum_{i,u,j,k} h_{ijk}^t h_{ijk}^t = 8 \sum_{a,b,c} k_{abc}^\alpha \bar{k}_{abc}^\alpha. \]

The Laplacian is given by

\[ \Delta = -4 \sum_a \nabla_a \nabla_a. \]

We define \( A_m \) by

\[ A_m = \sum_{\alpha,a_1,\ldots,a_m} k_{a_1\cdots a_m}^\alpha \bar{k}_{a_1\cdots a_m}^\alpha. \]

Now, we say that the immersion is of degree \( m_0 \) if there exists a positive integer \( m_0 \) in such a way that \( A_{m_0} \neq 0, A_{m_0+1} = 0 \). We need the following.

**Lemma 5** ([11]). Let \( f_{p_i} : M_i \to CP^m \) be the \( p_i \)-th full Kaehler imbedding of a compact irreducible Hermitian symmetric space \( M_i \) of rank \( r_i \), and let \( f \) be the tensor product of \( f_{p_i} \) (\( i=1, \ldots, s \)). Then the degree of \( f \) is \( \sum_{i=1}^s p_i r_i \).

If \( M \) is an \( n \)-dimensional locally symmetric Einstein Kaehler submanifold with \( T=kg \) (see, **Proposition 1**), then we have

\[ \sum_a k_{a0c}^\alpha \bar{k}_{d}^\alpha = 0 \quad \text{and} \quad \sum_{a,b} k_{a0b}^\alpha \bar{k}_{ab}^\alpha = 0, \]

so that from (6.14) we get

\[ \sum_d k_{abcd}^\alpha = \left( \frac{n+3}{2} - \frac{3\|\sigma\|^2}{4n} \right) k_{ab}^\alpha. \]

Hence if \( M \) is an \( n \)-dimensional locally symmetric Einstein Kaehler submanifold with \( T=kg \), \( A_4 = 0 \) and \( \tau \neq n(n-3)/3 \), then \( \nabla \sigma = 0 \). Therefore, from **Proposition 1**, **Lemma 5** and Table 2, we see that \( CP^n(1/3) \) and compact irreducible Hermitian symmetric spaces of rank 3 cannot be of order \( \{k_1, k_2\} \). Consequently, from **Lemma 3** and Table 2 we have the following.

**Proposition 5.** Compact irreducible Hermitian symmetric submanifolds of degree 3 are of order \( \{1, 2, 3\} \).

**§ 7. Proof of Theorem.**

Let \( R, S, \tau, T, \sigma \) be the curvature tensor, the Ricci tensor, the scalar curvature, the tensor given in \([3.5]\) and the second fundamental form of \( M \) respectively, and let \( \bar{R}, \bar{S}, \bar{\tau}, \bar{T} \) and \( \bar{\sigma} \) be the ones of \( \bar{M} \). First, we get (see [11])

\[ \dim(M) = \dim(\bar{M}), \quad \text{vol}(M) = \text{vol}(\bar{M}), \quad \int_M \tau = \int_{\bar{M}} \bar{\tau}. \]
and
\[ \int_M (2\|R\|^2 - 2\|S\|^2 + 5\tau^2) = \int_{\tilde{M}} (2\|\tilde{R}\|^2 - 2\|\tilde{S}\|^2 + 5\tilde{\tau}^2). \]
These, together with the fact that \( M \) and \( \tilde{M} \) are Einstein, yield
\[ \tau = \tilde{\tau}, \quad \|S\|^2 = \|\tilde{S}\|^2 \quad \text{and} \quad \int_M \|R\|^2 = \int_{\tilde{M}} \|\tilde{R}\|^2. \]
Then, from (5.13), (5.15) and (5.16) we see that
\[(7.1) \quad \|\sigma\|^2 = \|\tilde{\sigma}\|^2, \quad \int_M \|T\|^2 = \int_{\tilde{M}} \|\tilde{T}\|^2 \quad \text{and} \quad \int_M \|\nabla\sigma\|^2 = \int_{\tilde{M}} \|\nabla\tilde{\sigma}\|^2. \]
Moreover, since \( M \) is Einstein,
\[(7.2) \quad \int_M \left(-\frac{1}{9}\|\nabla R\|^2 + \frac{8}{21}\sum R_{ijkl}^* R_{klmn}^* R_{mnij}^* \right) \]
is a spectral invariant (see [10]), where \( R_{ijkl}^* \) denotes the components of \( R \) with respect to the real local orthonormal frames.

We see that
\[(7.3) \quad \sum R_{ijkl}^* R_{klmn}^* R_{mnij}^* = 64 \sum R_{abc} R_{def} R_{fa b}. \]
From (6.9) we get
\[ \sum R_{abc} R_{def} R_{fa b} = \frac{n(n+1)(n+3)}{64} \left( \|\sigma\|^2 + \frac{\|T\|^2}{32} + \frac{n+3}{32n} \right) \]
\[ - \sum k_{abc}^\alpha k_{a c}^\beta \delta_{ab} k_{a b}^\alpha k_{a b}^\beta \delta_{ab} k_{a b}^\alpha k_{a b}^\beta \delta_{ab} \]
This, together with (7.1)~(7.3), implies that
\[(7.4) \quad \int_M \left(-\frac{1}{9}\|\nabla R\|^2 + \frac{512}{21}\sum k_{abc}^\alpha k_{a c}^\beta \delta_{ab} k_{a b}^\alpha k_{a b}^\beta \delta_{ab} \right) \]
is a spectral invariant. From Lemma 4, we have
\[ k_{abc}^\alpha = \sum_{\alpha,b,c} k_{abc}^\alpha = \frac{n+1}{4} \left( k_{a b}^\alpha k_{a c}^\beta + k_{a b}^\beta k_{a c}^\gamma + k_{a c}^\alpha k_{a b}^\beta \right) \]
from which it follows that
\[ \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha k_{a c}^\alpha = 3 \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha k_{a c}^\alpha - 3 \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha k_{a c}^\alpha. \]
Since \( M \) is Einstein, we have
\[ \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha k_{a c}^\alpha = \left( \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha k_{a c}^\alpha \right)_{c} = 0, \]
so that we get
\[(7.5) \quad \sum_{\alpha,b,c} k_{abc}^\alpha k_{a b}^\alpha (k_{a b c})_{d} = 0. \]
Then it follows that

$$\frac{-1}{4}\Delta A_3 = \sum_{\alpha, \beta, \gamma, \delta} (k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a) d\bar{a}$$

$$= \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a + \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a$$

$$= \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a + A_4.$$

Hence we obtain

$$\int_M A_4 = -\int_M \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a.$$

Then, from Lemma 4, we see that

(7.6) \[ \int_M A_4 = \int_M \left\{ -\left[ (n+3)/2 - 3\|\sigma\|^2/4n \right] A_3 + \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a - A_4 \right\} \]

On the other hand, from (7.5) we get

\[
0 = \sum (k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a) d\bar{a} = \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a + \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a d\bar{a}
\]

\[
= \frac{3(n+2)\|\sigma\|^2}{64} - \frac{3}{16} \left( \|T\|^2 + \frac{\|\sigma\|^2\|T\|^2}{n} \right) + \frac{3\|\sigma\|^2\|T\|^2}{32n} + \frac{3}{4} A_3 - 3\sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a
\]

from which it follows that

(7.7) \[ \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a = \frac{1}{2} \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a + \text{term of } \{n, \|\sigma\|^2, \|T\|^2, A_3\}. \]

This, together with (7.6), implies

(7.8) \[ \int_M \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a = \int_M \left( \frac{1}{6} A_4 - \frac{1}{6} \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a \right) + \text{term of } \{n, \|\sigma\|^2, \|T\|^2, A_3\}. \]

Therefore, from (7.4) and (7.8) we see that

$$\int_M \left( -\frac{1}{9} \|\nabla R\|^2 + \frac{256}{63} \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a - \frac{256}{63} A_4 \right)$$

is a spectral invariant. On the other hand, from (6.15) and (6.16) we get

$$\|\nabla R\|^2 = 64 \sum k_{\alpha\beta\gamma\delta}^a \bar{k}_{\alpha\beta\gamma\delta}^a k_{\alpha\beta\gamma\delta}^a,$$

from which it follows that
Table 2. Compact irreducible Hermitian symmetric submanifolds of degree 3.

<table>
<thead>
<tr>
<th>submanifold</th>
<th>dim&lt;sub&gt;c&lt;/sub&gt;</th>
<th>p</th>
<th>∥S∥²</th>
<th>∥R∥²</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP&lt;sup&gt;n&lt;/sup&gt;(1/3)</td>
<td>n</td>
<td>n(n+1)(n+5)/6</td>
<td>n(n+1)&lt;sup&gt;2&lt;/sup&gt;/18</td>
<td>2n(n+1)/9</td>
</tr>
<tr>
<td>SU(r+3)/SU(U(r)×U(3))&lt;sup&gt;(r≥3)&lt;/sup&gt;</td>
<td>3r</td>
<td>r(r-1)(r+7)/6</td>
<td>3r(r+3)&lt;sup&gt;2&lt;/sup&gt;/2</td>
<td>6r(3r+1)</td>
</tr>
<tr>
<td>Sp(3)/U(3)</td>
<td>6</td>
<td>7</td>
<td>48</td>
<td>66</td>
</tr>
<tr>
<td>SO(12)/U(6)</td>
<td>15</td>
<td>16</td>
<td>750</td>
<td>660</td>
</tr>
<tr>
<td>SO(14)/U(7)</td>
<td>21</td>
<td>42</td>
<td>1512</td>
<td>1344</td>
</tr>
<tr>
<td>E₄/E₅×T</td>
<td>27</td>
<td>28</td>
<td>4374</td>
<td>3132</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>τ</th>
<th>∥σ∥²</th>
<th>∥T∥²</th>
<th>μ</th>
<th>λ₁</th>
<th>λ₂</th>
<th>λ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>n(n+1)/3</td>
<td>2n(n+1)/3</td>
<td>4n(n+1)/9</td>
<td>1/3</td>
<td>(n+1)/3</td>
<td>2(n+2)/3</td>
<td>n+3</td>
</tr>
<tr>
<td>3r(r+3)</td>
<td>6r(r-1)</td>
<td>12r(r-1)</td>
<td>-1</td>
<td>r+3</td>
<td>2r+4</td>
<td>3r+3</td>
</tr>
<tr>
<td>24</td>
<td>18</td>
<td>27</td>
<td>-1/2</td>
<td>4</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>150</td>
<td>90</td>
<td>270</td>
<td>-2</td>
<td>10</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>252</td>
<td>210</td>
<td>630</td>
<td>-2</td>
<td>12</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>486</td>
<td>270</td>
<td>1350</td>
<td>-4</td>
<td>18</td>
<td>28</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 3. Compact reducible Einstein Hermitian symmetric submanifolds of degree 3.

<table>
<thead>
<tr>
<th>submanifold</th>
<th>dim&lt;sub&gt;c&lt;/sub&gt;</th>
<th>λ₁</th>
<th>λ₂</th>
<th>λ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP&lt;sup&gt;n&lt;/sup&gt;×CP&lt;sup&gt;n&lt;/sup&gt;×CP&lt;sup&gt;n&lt;/sup&gt;</td>
<td>3n</td>
<td>n+1</td>
<td>2n+2</td>
<td>2n+4</td>
</tr>
<tr>
<td>CP&lt;sup&gt;n&lt;/sup&gt;×CP&lt;sup&gt;2n+1&lt;/sup&gt;(1/2)</td>
<td>n+1</td>
<td>n+1</td>
<td>2n+2</td>
<td>2n+3</td>
</tr>
<tr>
<td>CP&lt;sup&gt;n&lt;/sup&gt;×Q&lt;sup&gt;n+1&lt;/sup&gt;&lt;sup&gt;(n≥2)&lt;/sup&gt;</td>
<td>2n+1</td>
<td>n+1</td>
<td>n+3</td>
<td>2n+2</td>
</tr>
<tr>
<td>CP&lt;sup&gt;n&lt;/sup&gt;×{SU(n+1)/SU(U(2)×U(n-1))]&lt;sup&gt;(n≥4)&lt;/sup&gt;</td>
<td>3n-2</td>
<td>n+1</td>
<td>2n</td>
<td>2n+2</td>
</tr>
<tr>
<td>CP₁×{SO(10)/U(5)}</td>
<td>17</td>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>CP&lt;sup&gt;11&lt;/sup&gt;×{E₆/Spin(10)×T}</td>
<td>27</td>
<td>12</td>
<td>18</td>
<td>24</td>
</tr>
</tbody>
</table>
\begin{equation*}
\int_M (3\|\nabla R\|^2 + 256A_4)\end{equation*}

is a spectral invariant. Since $\tilde{M}$ is locally symmetric and of degree 3, it follows that $M$ is also locally symmetric and of degree $\leq 3$. Hence $M$ is a compact Hermitian symmetric submanifold of degree $\leq 3$. From Lemma 3, Proposition 2, Tables 1~3 and Theorem 4.3 in [6], $M$ is one of the compact Hermitian symmetric submanifolds given in Tables 2 and 3.

Q. E. D.

Eigenvalues for classical symmetric spaces (up to their ranks) are computed by T. Nagano [5] and eigenvalues for exceptional types are computed in § 4, and eigenvalues for ones given in Table 3 can be computed in the same way. And from Lemma 2.4 in [6], we get

\begin{equation*}
\|T\|^2 = (1-\mu)\|\sigma\|^2,
\end{equation*}

where $\mu$ is given in Table 2. Since the scalar curvatures for irreducible Hermitian symmetric spaces are given in Table 2 of [6], from the above formula and (5.13)~(5.16) we can compute the values of $\|\sigma\|^2, \|T\|^2, \|S\|^2$ and $\|R\|^2$.

References


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