Minimal Immersions of Kaehler Manifolds into Complex Space Forms

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(Communicated by K. Ogiue)

Introduction

Let $N^m(c\sim)$ be an $m$-dimensional complex space form of constant holomorphic sectional curvature $c\sim$, and let $(M^n, g)$ be an $n$-dimensional Kaehler manifold. It is well known that holomorphic isometric immersions of Kaehler manifolds into Kaehler manifolds are minimal immersions. We consider the following problem: Is an isometric minimal immersion $f: (M^n, g) \to N^m(c\sim)$ a holomorphic or anti-holomorphic immersion? However, it is not true in general. For example, if we take $M^1=RH^2(c\sim/4)$ (2-dimensional real hyperbolic space of constant curvature $c\sim/4$, $c\sim<0$) or $M^1=S^2(c\sim/4)$ (2-dimensional sphere of constant curvature $c\sim/4$, $c\sim>0$), then we obtain totally real isometric minimal immersions as follows:

(1) $M^1=RH^2(c\sim/4)$ totally real, totally geodesic $\to CH^m(c\sim)$,

where $CH^m(c\sim)$ is an $m$-dimensional complex hyperbolic space of constant holomorphic sectional curvature $c\sim$.

(2) $M^1=S^2(c\sim/4)$ natural covering $\to RP^2(c\sim/4)$

totally real, totally geodesic $\to CP^m(c\sim)$,

where $CP^m(c\sim)$ is an $m$-dimensional complex projective space of constant holomorphic sectional curvature $c\sim$ and $RP^2(c\sim/4)$ is a 2-dimensional real projective space of constant curvature $c\sim/4$.

In Part I, we prove the following

Theorem 1. Let $CH^m(c\sim)$ be an $m$-dimensional complex hyperbolic space of constant holomorphic sectional curvature $c\sim (c\sim<0)$, and let $(M^n, g)$ be an $n$-dimensional Kaehler manifold such that $\dim_c M=n\geq 2$. Then,
Every minimal isometric immersion of \((M^n, g)\) into \(CH^n(\bar{c})\) is a holomorphic or anti-holomorphic immersion.

**Remark 1.** By (1), Theorem 1 is not true if \(\dim_c M = 1\). Our result is closely related to Siu’s work [5]. In fact, by using Siu’s arguments we can show that Theorem 1 holds even if \(CH^n(\bar{c})\) is replaced by a Kaehler manifold of strongly negative curvature tensor. On the other hand, we can prove Theorem 1 by using Gauss equation only. \(CH^n(\bar{c})\) and its quotient manifolds are the examples of Kaehler manifolds of strongly negative curvature tensor (for example, see [5], [6]). Since Theorem 1 is of local nature, we can easily see that Theorem 1 holds when \(CH^n(\bar{c})\) is replaced by its quotient manifolds.

In Part II, we prove the following theorem, which is some generalization of Micallef’s result [4].

**Theorem 2.** Let \(f: M^n \rightarrow R^{2n+2}/D\) be an isometric stable minimal immersion of an \(n\)-dimensional compact Kaehler manifold into a \((2n+2)\)-dimensional flat torus. Assume that \(|R|^2 \geq \tau^2\) holds on \(M\), where \(R\) is the curvature tensor and \(\tau\) is the scalar curvature of \(M\). Then, \(f\) is holomorphic with respect to some orthogonal complex structure of \(R^{2n+2}/D\).

**Remark 2.** The case of \(n=1\) in Theorem 2 is proved by Micallef ([4], p. 73, Theorem 1’), because the condition \(|R|^2 \geq \tau^2\) is automatically satisfied when \(n=1\). For every complex hypersurface in \(C^{n+1}/D\), \(|R|^2 = \tau^2\) holds.

The author wishes to thank Professors K. Ogiue, N. Ejiri and H. Tsuji for many valuable comments and suggestions.

**Part I. The Case of \(\bar{c} < 0\)**

§1. Proof of Theorem 1.

Let \(f: (M^n, g) \rightarrow (N^n(\bar{c}), h)\) be an isometric minimal immersion of an \(n\)-dimensional Kaehler manifold into an \(m\)-dimensional complex space form of constant holomorphic sectional curvature \(\bar{c}\). Let \(TM^c\) (resp. \(TN^c\)) be the complexification of the tangent bundle of \(M\) (resp. \(N\)). Then, the differential mapping \(f_*: TM \rightarrow TN\) can be naturally extended to the complex linear mapping, which is also denoted by \(f_*\). We have the Gauss equation

\[
h(\tilde{R}(f_*(X), f_*(Y))f_*(Z), f_*(W))
\]

(3)
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\[ g(R(X, Y)Z, W) + h(\sigma(X, Z), \sigma(Y, W)) \]
\[ - h(\sigma(Y, Z), \sigma(X, W)) , \quad \text{for } X, Y, Z, W \in TM^c , \]

where \( R, \tilde{R} \) and \( \sigma \) denote the curvature tensor of \( M \), the curvature tensor of \( N \) and the second fundamental form of \( f \), respectively. They are also extended to the complex tensors. To prove Theorem 1, it is enough to show that \( f \) is a holomorphic or anti-holomorphic immersion at each point \( p \in M \). Therefore, we choose any point \( p \in M \) and we verify the assertion at \( p \). We choose unitary bases \( e_1, \cdots, e_n \) of \( T_p M^c \) and \( u_1, \cdots, u_m \) of \( T_{f(p)} N^c \). We use the following convention on the range of indices:

\[ \alpha, \beta, \gamma, \cdots = 1, \cdots, m \quad i, j, k, \cdots = 1, \cdots, n \]
\[ \lambda, \mu, \nu, \cdots = 1, \cdots, m, \overline{1}, \cdots, \overline{m} \]
\[ A, B, C, \cdots = 1, \cdots, n, \overline{1}, \cdots, \overline{n} . \]

Then, we have

(4) \[ h(\tilde{R}(u_{\alpha}, u_{\overline{\beta}})u_{\gamma}, u_{\overline{\delta}}) = \frac{c}{2} (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}) . \]

If we put

\[ f_*(e_i) = \sum_{\alpha} f_i^\alpha u_\alpha + \sum_{\alpha} f_{\overline{i}}^\alpha u_{\overline{\alpha}} , \]
\[ f_*(e_{\overline{i}}) = \sum_{\alpha} f_{\overline{i}}^\alpha u_\alpha + \sum_{\alpha} f_i^\alpha u_{\overline{\alpha}} , \]

then we have

(5) \[ \sum_{\alpha} f_i^\alpha f_{\overline{\alpha}} + \sum_{\alpha} f_{\overline{i}}^\alpha f_i^\alpha = \delta_{ij} \]
(6) \[ \sum_{\alpha} f_i^\alpha f_{\overline{\alpha}} + \sum_{\alpha} f_{\overline{i}}^\alpha f_i^\alpha = 0 \quad \text{for any } i \text{ and } j , \]

because both \( g \) and \( h \) are Kaehler metrics. Moreover, since \( M \) and \( N \) are Kaehler manifolds, we have

(7) \[ g(R(e_A, e_B) e_i, e_j) = g(R(e_A, e_B) e_i, e_j) \]
\[ = g(R(e_i, e_j) e_A, e_B) = 0 , \]
\[ h(\tilde{R}(u_\alpha, u_{\overline{\beta}}) u_{\gamma}, u_{\overline{\delta}}) = h(\tilde{R}(u_{\alpha}, u_{\overline{\beta}}) u_{\gamma}, u_{\overline{\delta}}) \]
\[ = h(\tilde{R}(u_\alpha, u_{\overline{\beta}}) u_{\gamma}, u_{\overline{\delta}}) = 0 \]
\[ \text{for any } A, B, i, j, \lambda, \mu, \alpha \text{ and } \beta . \]

Therefore, if we put \( X=e_i, Y=e_j, Z=e_{\overline{j}} \) and \( W=e_{\overline{i}} \) in (3), it follows
from (4), (6) and (7) that

\[
2\sigma \sum_{a,i,j} f_i^a f_i^a f_j^\alpha f_j^\alpha + \bar{\sigma} \sum_{a,i,j} \left( f_i^a f_i^a f_j^\alpha f_j^\alpha - f_i^\alpha f_i^\alpha f_j^a f_j^a \right) = \sum_{i,j} h(\sigma(e_i, e_j), \sigma(e_i, e_j)) - \sum_{i,j} h(\sigma(e_i, e_j) \cdot \sigma(e_i, e_j)) .
\]

Since \( f \) is minimal, we have \( \sum_i \sigma(e_i, e_i) = 0 \), which implies that the right hand side of (8) is non-negative, and so is the left hand side of (8). We claim that

\[
\sum f^\alpha f^\underline{\alpha} f_i^\alpha f_j^\underline{\alpha} \geq \sum f^\alpha f^\underline{\alpha} f_i^\alpha f_j^\underline{\alpha} .
\]

In fact, since \( \sum_a f_i^a f_j^a \) is a Hermitian matrix, by choosing a suitable unitary basis, we can assume that

\[
\sum_a f_i^a f_j^a = \sum_a f_i^a f_i^a (\geq 0) \quad \text{if } i = j
\]

\[= 0 \quad \text{if } i \neq j ,
\]

which, by (5), implies that

\[
\sum_a f_i^a f_j^a = \sum_a f_i^a f_i^a (\geq 0) \quad \text{if } i = j
\]

\[= 0 \quad \text{if } i \neq j .
\]

Then, the claim (9) is proved. If \( \sigma < 0 \), the first term of the left hand side of (8) is non-positive and the second term is also non-positive by (9). On the other hand, since the right hand side of (8) is non-negative, it follows that the equality in (9) holds. If we put \( \lambda_i = \sum_a f_i^a f_i^a \) and \( \mu_i = \sum_a f_i^a f_i^a \), then from (9), (10) and (11) we have \( \lambda_i \mu_i = 0 \) for \( 1 \leq i \neq j \leq n \). By \( n \geq 2 \) and (5) we obtain \( \lambda_i = 0 \) for any \( i \) or \( \mu_i = 0 \) for any \( i \), which implies that \( f \) is holomorphic or anti-holomorphic. Therefore, Theorem 1 is proved.

\section{Some remarks on Section 1.}

If \( \sigma = 0 \), from (8) we obtain

\[
\sigma(e_i, e_j) = 0 \quad \text{for any } i \text{ and } j ,
\]

which is equivalent to

\[
f_i^a = 0 \quad \text{for any } \alpha, i \text{ and } j .
\]

It is well known that an isometric immersion \( f \) is minimal if and only if \( f \) is a harmonic map. The mapping \( f \) satisfying (13) is called "pluri-
harmonic”. We remark the following

**PROPOSITION 1.** Let $f: (M^n, g) \to N^m(0)$ be an isometric minimal immersion of a Kaehler manifold. Then, $f$ is pluriharmonic.

**PROPOSITION 2.** Let $f: (M^n, g) \to \mathbb{C}P^n$ be an isometric immersion of a Kaehler manifold into a complex projective space with the Fubini-Study metric. If $n \geq 2$ and $f$ is pluriharmonic, then $f$ is holomorphic or antiholomorphic.

**REMARK 3.** Proposition 1 is already obtained in [1], p. 212, Theorem 1.2. Moreover, we remark that a submanifold satisfying (12) is called *austere* by R. Harvey and H. B. Lawson, Jr. ([3], p. 102).

**PART II. THE CASE OF $\bar{c}=0$.**

First, we state the following

$(\ast)$ Let $f: M^n \to \mathbb{R}^{2n+2}$ be an isometric stable minimal immersion of an $n$-dimensional complete Kaehler manifold into a $(2n+2)$-dimensional Euclidean space. Assume that $M$ is parabolic, that is, $M$ admits no positive non-constant superharmonic functions and that $|R|^2 \geq \tau^2$ holds everywhere on $M$. Then, $f$ is holomorphic with respect to some orthogonal complex structure of $\mathbb{R}^{2n+2}$.

The case of $n=1$ in $(\ast)$ is proved in [4]. We prove $(\ast)$ by generalizing the method of [4] to Kaehler manifolds. Theorem 2 can be proved immediately by the same method as the proof of $(\ast)$. Unfortunately, for the case of $n \geq 2$, we know no examples of parabolic Kaehler manifolds.

§1. Stability condition.

Let $f: M^n \to \mathbb{R}^{2n+2}$ be an isometric stable minimal immersion of an $n$-dimensional Kaehler manifold into a $(2n+2)$-dimensional Euclidean space. We define the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$ by

\[(1.1) \quad \langle s, t \rangle = s \cdot \overline{t} \quad \text{for any } s, t \in \mathbb{C}^{2n+2}, \]

where

\[s \cdot t = \sum_{i=1}^{2n+2} s_i t_i \quad \text{for } s = (s_1, \ldots, s_{2n+2}) \text{ and } t = (t_1, \ldots, t_{2n+2}). \]

Let $TM^c$ be the complexification of the tangent bundle $TM$ of $M$. Then,
we have $TM^c = TM^{1.0} + TM^{0.1}$, where the fibre $T_p M^{1.0}$ (resp. $T_p M^{0.1}$) at $p \in M$ is the $\sqrt{-1}$-eigenspace (resp. $-\sqrt{-1}$-eigenspace) of the complex structure tensor of $M$. Let $NM$ be the normal bundle of $M$. Then, naturally $NM$ has a complex structure defined by its orientation (i.e., rotation by 90°), and with respect to this complex structure, we have

$$NM^c = NM^{1.0} + NM^{0.1}.$$ 

We denote by $C^\infty_0(NM^c)$ the set of all compactly supported smooth sections of $NM^c$. Then, we have the stability inequality for a minimal submanifold of $C^{n+1}$ (see [4])

$$(1.2) \quad \int_X |(ds)^T|^2 \leq \int_X |(ds)^N|^2$$

for any $s \in C^\infty_0(NM^c)$, where $d$ is the Riemannian connection of $C^{n+1}$, and the superscripts $T$ and $N$ denote orthogonal projections onto the tangent space and normal space of $M$ respectively. Let $(x^i)$ $(i=1, \cdots, n)$ be a local complex coordinate system on $M$. Then, we can write $ds=\partial s + \overline{\partial} s$, where $\partial s = \sum_{i} (\partial_i s) dz^i$ and $\overline{\partial} s = \sum_{i} (\partial - s) d\overline{z}^i$. Thus, (1.2) can be rewritten as

$$(1.3) \quad 2 \int_X |(ds)^T|^2 \leq \int_X |ds|^2 = \int_X |\partial s|^2 + \int_X |\overline{\partial} s|^2.$$ 

Since the connection $d$ is flat and $s$ has compact support, by the integration by parts we have

$$\int_X |\partial s|^2 = \int_X |\overline{\partial} s|^2.$$ 

This, together with (1.3), yields

$$(1.4) \quad \int_X |(\partial s)^T|^2 \leq \int_X |(\overline{\partial} s)^N|^2,$$

or

$$(1.5) \quad \int_X |(\overline{\partial} s)^T|^2 \leq \int_X |(\partial s)^N|^2.$$ 

§2. A condition for $f$ to be holomorphic.

Micallef [4] proved the following

**THEOREM A.** Let $F: M^n \rightarrow \mathbb{R}^{2n}$ be an immersion of an $n$-dimensional complex manifold into an $2n$-real dimensional Euclidean space with the
usual metric. Assume that there exist vector bundles $E$ and $V$ over $M$ which satisfy the following conditions:

i) $TM^c = E \oplus \overline{E}$, $NM^c = V \oplus \overline{V}$,

ii) $E \oplus V$ is orthogonal to $E \oplus \overline{V}$ with respect to $\langle \cdot, \cdot \rangle$,

iii) $d : \Gamma(E \oplus V) \rightarrow \Gamma((E \oplus V) \otimes T^*M)$.

Then, there exist complex structures $\tilde{J}$ and $J$ on $M$ and $R^{2n}$ respectively such that $\tilde{J}$ is orthogonal with respect to the metric induced on $M$ by $F$, $J$ is orthogonal with respect to the Euclidean inner product on $R^{2n}$ and $F$ is holomorphic with respect to $\tilde{J}$ and $J$.

REMARK 4. $J$ is actually covariant constant on $R^{2n}$, so that $R^{2n}$ is endowed with the usual Kaehler structure of $C^n$.

We can apply Theorem A to our situation. We put $E = TM^{1,0}$ and $V = NM^{1,0}$. Then, the conditions i) and ii) in Theorem A are satisfied. To see the condition iii) in Theorem A more precisely, we choose local fields of unitary frames $e_1, \ldots, e_n$ and $e_{n+1}$ for $TM^c$ and $NM^c$, respectively. Then,

$$de_i = \sum_{j=1}^{n} \omega_{ij} \otimes e_j + [de_i]^N \quad \text{for} \quad i = 1, \ldots, n,$$

and

$$de_{n+1} = \omega_{n+1,\overline{n}+1} \otimes e_{n+1} + [de_{n+1}]^T,$$

where $\omega_{ij}$ and $\omega_{n+1,\overline{n}+1}$ are the connection 1-forms for $TM^c$ and $NM^c$ respectively. Therefore, the condition iii) in Theorem A is satisfied if and only if

$$[de_{n+1}] \cdot e_i \equiv 0 \quad \text{for} \quad i = 1, \ldots, n. \quad (2.1)$$

We denote by $f_A$, $f_{AB}$ and $f_{ABC}$, the first order covariant derivative with respect to $e_A$, the second order covariant derivative with respect to $e_A$ and $e_B$, the third order covariant derivative with respect to $e_A$, $e_B$ and $e_C$ of an immersion $f$, respectively. From now on, we use the following convention on the range of indices:

$$i, j, k, \ldots = 1, \ldots, n; \quad A, B, C, \ldots = 1, \ldots, n, \overline{1}, \ldots, \overline{n}.$$

Then, we can easily see that (2.1) is equivalent to

$$f_{ij}^{0,1} \equiv 0, \quad (f_{ij})^{0,1} \equiv 0 \quad \text{for any} \quad i \text{ and } j, \quad (2.2)$$

where $(f_{AB})^{0,1}$ is the $NM^{0,1}$-component of $f_{AB}$. Note that (2.2) is also
equivalent to

\[ (\partial s)^r \equiv 0 \quad \text{for any} \quad s \in C^0_0(NM^{1,0}). \]

\section{3. Proof of (\star).}

From Proposition 1, we already know that \( f \) satisfies

\[ (f_i^j)^{*,1} \equiv 0 \quad \text{for any} \quad i \text{ and} \quad j. \]

\textbf{Lemma 3.1.} For a fixed vector \( a \in C^{2n+2} \), we have

\[ \sum_i D_i D_i a^{1,0} = -\sum_{i,j} (a \cdot f_{ij})(e_{n+1} \cdot f_{ij})e_{n+1}, \]

where \( D \) is the normal connection of \( M \) and \( a^{1,0} \) is the \( NM^{1,0} \) component of \( a \).

\textbf{Proof.} First, note that \( f_{AB} \in NM^c \). We may write \( a^{1,0} = (a \cdot e_{n+1})e_{n+1} \). Using (3.1), we see that

\[ (\partial e_{n+1})^r = -\sum_j (e_{n+1} \cdot f_{ij})f_j, \]

\[ (\partial e_{n+1})^N = De_{n+1} = \omega_{n+1,n+1}(e) e_{n+1}. \]

Then, we have

\[
D_i a^{1,0} = (a \cdot (\partial e_{n+1})^r)e_{n+1} + (a \cdot D_i e_{n+1})e_{n+1} \\
+ (a \cdot e_{n+1})D_i e_{n+1} \\
= -\sum_j (a \cdot f_j)(e_{n+1} \cdot f_{ij})e_{n+1} \\
+ \omega_{n+1,n+1}(e_i)(a \cdot e_{n+1})e_{n+1} + \omega_{n+1,n+1}(e_i)(a \cdot e_{n+1})e_{n+1} \\
= -\sum_j (a \cdot f_j)(e_{n+1} \cdot f_{ij})e_{n+1}.
\]

Moreover, we have

\[ \sum_i D_i D_i a^{1,0} = -\sum_{i,j} (a \cdot f_{ij})(e_{n+1} \cdot f_{ij})e_{n+1} \\
- \sum_{i,j} (a \cdot f_j)((\partial e_{n+1})^N \cdot f_{ij}) + (e_{n+1} \cdot f_{ij})e_{n+1} \\
- \sum_{i,j} (a \cdot f_j)(e_{n+1} \cdot f_{ij})D_i e_{n+1}. \]

Since \( (\sum_i f_{ij})^N = (\sum_i f_{ij})^N = 0 \) by Ricci identity and the minimality of \( f \), (3.5), together with (3.4) and the fact that \( De_{n+1} = \omega_{n+1,n+1}(e_i)e_{n+1} \), yields (3.2).

Q.E.D.
We put $s = \lambda \sigma$ in the stability inequality (1.4), where $\lambda$ is a smooth $R$-valued function with compact support and $\sigma \in C^\infty(NM^c)$. Then, we may rewrite (1.4) as

\begin{equation}
\int_M \lambda^2 \sum_i |(\partial_i \sigma)^\tau|^2 \leq \int_M |\lambda_i|^2 |\sigma|^2 + \int_M \sum_i (\lambda \lambda_i (\sigma \cdot D_i \overline{\sigma}) + \lambda \lambda_i (\overline{\sigma} \cdot D_i \sigma)) + \int_M \lambda^2 \sum_i (D_i \overline{\sigma} \cdot D_i \sigma).
\end{equation}

Since

\begin{align*}
0 &= \int_M \sum_i \{ \partial_i (\lambda^2 (\sigma \cdot D_i \overline{\sigma})/2) + \partial_i (\lambda^2 (\overline{\sigma} \cdot D_i \sigma)/2) \} \\
&= \int_M \sum_i \{ \lambda \lambda_i (\sigma \cdot D_i \overline{\sigma}) + \lambda \lambda_i (\overline{\sigma} \cdot D_i \sigma) \} \\
&\quad + \int_M \sum_i \lambda^2 (D_i \overline{\sigma} \cdot D_i \sigma) + \int_M \sum_i \lambda^2 (\sigma \cdot D_i \overline{\sigma}) \\
&\quad + \int_M \sum_i \lambda^2 (D_i \overline{\sigma} \cdot D_i \sigma),
\end{align*}

(3.6) may be rewritten as

\begin{equation}
\int_M \sum_i \lambda^2 |(\partial_i \sigma)^\tau|^2 \leq \int_M \sum_i |\lambda_i|^2 |\sigma|^2 - \int_M \sum_i \lambda^2 \text{Re}((\overline{\sigma} \cdot D_i D_i \sigma)) + \int_M \sum_i \lambda^2 |(\partial_i \sigma)^\tau|^2.
\end{equation}

Note that

\begin{align*}
\sum_i |(\partial_i \sigma)^\tau|^2 &= \sum_i |\sigma \cdot f_{ij}|^2 \quad \text{and} \quad |d\lambda|^2 = 2 \sum_i |\lambda_i|^2,
\end{align*}

which, together with (3.7), yield

\begin{equation}
2 \int_M \lambda^2 \sum_{i,j} |\sigma \cdot f_{ij}|^2 + 2 \int_M \lambda^2 \sum_i \text{Re}((\overline{\sigma} \cdot D_i D_i \sigma)) \\
\leq \int_M |d\lambda|^2 |\sigma|^2.
\end{equation}

For a unit vector $a \in C^{2n+2}$, we put $\sigma = a^{1,0}$ in (3.8). Then, by Lemma 3.1, we obtain

\begin{equation}
\int_M \lambda^2 q \leq \int_M |d\lambda|^2 |a^{1,0}|^2 \leq \int_M |d\lambda|^2,
\end{equation}

where

\begin{equation}
q = -2 \text{Re} \left( \sum_{i,j} (a \cdot (f_{ij})^{1,0})(\overline{a} \cdot (f_{ij})^{1,0}) \right).
\end{equation}

Since (3.9) holds for all smooth functions $\lambda$ with compact support, using a theorem of D. Fisher-Colbrie and R. Schoen [2], we see that there
exists a smooth function $u>0$ on $M$ such that

(3.11) \[-\Delta u + qu = 0,\]

where

$$\Delta u = -2 \sum_{i} u_{ii}.$$ 

If we put $w = \log u$, then we have

(3.12) \[\Delta w = q + |dw|^2.\]

Since

$$0 = \int_{M} \sum_{i} \{\partial_{\tau}(\lambda^{2} w_{i}) + \partial_{i}(\lambda^{2} w_{i})\} = 2 \int_{M} \lambda (d\lambda \cdot dw) - \int_{M} \lambda^{2} \Delta w,$$

from (3.12) we obtain

(3.13) \[2 \int_{M} \lambda (d\lambda \cdot dw) = \int_{M} \lambda^{2} q + \int_{M} \lambda^{2} |dw|^2.\]

Moreover, using $2|\lambda (d\lambda \cdot dw)| \leq (\lambda^{2}/2)|dw|^2 + 2|d\lambda|^2$, we have

(3.14) \[2 \int_{M} |d\lambda|^2 \geq \int_{M} \lambda^{2} q + \frac{1}{2} \int_{M} \lambda^{2} |dw|^2.\]

We choose a unitary basis $a_{1}, a_{2}, \ldots, a_{2n+2}$ of $C^{2n+2}$ and we let $q_{\alpha}$ stand for the expression in (3.10) with $a = a_{\alpha}$, $\alpha = 1, 2, \ldots, 2n+2$. The solution of (3.11) with $q = q_{\alpha}$ is denoted by $u_{\alpha}$ and $w_{\alpha} = \log u_{\alpha}$. Since we can easily see that

$$\sum_{\alpha=1}^{2n+2} q_{\alpha} = -2 \text{Re}\{\sum_{i,j} ((f_{ij})^{1,0} \cdot (f_{ij})^{1,0})\} = 0,$$

from (3.14) we have

(3.15) \[\int_{M} |d\lambda|^2 \geq \int_{M} \lambda^{2} r,\]

where

(3.16) \[r = \frac{1}{8(n+1)} \sum_{\alpha=1}^{2n+2} |dw_{\alpha}|^2.\]

Again by the theorem of Fischer-Colbrie and Schoen, we see that there
exists a smooth function \( v > 0 \) on \( M \) such that

\[-\Delta v + rv = 0.\]

Since \( r \geq 0 \), we have

\[\Delta v = rv \geq 0,\]

which implies that \( v \) is a positive superharmonic function. It follows from the parabolicity of \( M \) that \( v = \text{constant} \). Therefore, we have \( r \equiv 0 \) and \( w_\alpha = \text{constant} \), \( u_\alpha = \text{constant} \), for \( \alpha = 1, 2, \cdots, 2n + 2 \). It follows from (3.11) that

\[q_\alpha \equiv 0 \quad \text{for} \quad \alpha = 1, 2, \cdots, 2n + 2.\]

For any point \( p \in M \), either \( (f_{i\dot{f}})(p) = 0 \) for any \( i \) and \( j \), or \( (f_{i\dot{f}})(p) \neq 0 \) for some \( i \) and \( j \) holds.

Let \( L_p = \{(k, l)| (f_{kl})(p) \neq 0\} \) be the set of the pair of the indices \( k \) and \( l \) such that \( (f_{kl})(p) \neq 0 \) at \( p \in M \). Then, if we put \( a_i = (f_{i\dot{g}})(p)/|(f_{ij})(p)| \) for any \( (i, j) \in L_p \), it follows from \( q_i = 0 \) that

\[\text{Re}\{\sum_{k,l}((f_{ij})^{0,1}(p) \cdot (f_{kl})^{1,0}(p))((f_{\overline{i}\overline{j}})^{0,1}(p) \cdot (f_{\overline{k}\overline{l}})^{1,0}(p))\} = 0.\]

Since (3.17) holds for any \( (i, j) \in L_p \), we obtain

\[\text{Re}\{\sum_{i,j,k,l}((f_{ij})^{0,1} \cdot (f_{kl})^{1,0})(f_{\overline{i}\overline{j}})^{0,1} \cdot (f_{\overline{k}\overline{l}})^{1,0})\} = 0,\]

which implies

\[\sum_{i,j} f_{ij} f_{\overline{i}\overline{j}} = 0,\]

where \( f_{ij} \) is the component of a vector \( (f_{ij})^{1,0} \) with respect to \( e_{n+1} \). We also denote by \( f_{ij}^{\overline{1}} \) the component of a vector \( (f_{ij})^{0,1} \) with respect to \( e_{n+1} \).

From the Gauss equation we see that the curvature tensor \( R = (R_{ijkl}) \) and the scalar curvature \( \tau \) of \( M \) are given respectively by

\[R_{ijkl} = -f_{ik}^{\overline{1}} f_{jl}^{\overline{1}} - f_{ik} f_{jl}^{\overline{1}}, \]

\[\tau = 2 \sum_{i,j} R_{ij\overline{i}l}, \]

\[= -2 \sum_{i,j} \{|f_{ij}^{\overline{1}}|^2 + |f_{ij}^{\overline{1}}|^2\}.\]

Since \( |R|^2 = 4 \sum_{i,j,k,l} R_{ij\overline{i}l} R_{ij\overline{k}l} \), the assumption \( |R|^2 \geq \tau^2 \), together with (3.18) yields
$(\sum |f_{ij}^{1}|^{2})(\sum |f_{ki}^{1}|^{2})=0$ .

Therefore, we have proved that for any point $p \in M$, either

\begin{equation}
(3.19) \quad f_{ij}^{1}(p)=0 \quad \text{for any } i \text{ and } j ,
\end{equation}

or

\begin{equation}
\overline{f}_{ij}^{1}(p)=0 \quad \text{for any } i \text{ and } j
\end{equation}

holds.

We need the following

**Lemma 3.2.** $\sum_{i,j} (f_{ij}^{1})^{1,0} \otimes (dz dz^{\overline{i}})$ (resp. $\sum_{i,j} (f_{ij}^{1})^{0,1} \otimes (dz dz^{j})$) is a holomorphic quadratic differential with the value in $NM^{1,0}$ (resp. $NM^{0,1}$), where $(z^{i})$ is a local complex coordinate system on $M$.

**Proof.** Since $NM^{c}$ is the holomorphic vector bundle with the connection $D$ over $M$ and $D$ preserves $NM^{1,0}$ and $NM^{0,1}$, it is enough to show that

$$D_{\overline{k}}(f_{ij}^{1})=0 .$$

We see that

$$D_{\overline{k}}(f_{ij}^{1})=(\partial_{\overline{k}}f_{ij}^{1})^{N}=(f_{ikj}^{1})^{N}=(f_{ijk}^{1})^{N}=0 ,$$

because $f_{ij}=0$ for any $i$ and $j$.

Q.E.D.

Lemma 3.2, together with (3.19), implies that each of $f_{ij}^{1}$ and $\overline{f}_{ij}^{1}$ for any $i$ and $j$ vanishes either identically on $M$ or only at isolated points. Without loss of generality, we may assume that $f_{ij}^{1}$ vanishes identically for any $i$ and $j$.

Therefore, Theorem A, (2.2) and (3.1) imply that $f$ is holomorphic with respect to some orthogonal complex structure of $R^{2n+2}$. This completes the proof of $(\ast)$.

§4. Proof of Theorem 2.

By Remark in [4], p. 63, Theorem A still holds even if we replace $R^{2n+2}$ by $R^{2n+2}/D$. Moreover, by the same reason as Theorem 1' in [4, p. 73], (3.9) is still valid for any $\lambda \in C^{\infty}(M)$. Hence, we see that the lowest eigenvalue $\mu_{1}$ of $\Delta-q$ on $M$ is non-negative. Therefore, it follows that if $u>0$ is the eigenfunction of $\Delta-q$ corresponding to $\mu_{1}$, then we get $\Delta u-qu \geqq 0$. If we put $w=\log u$, then we have $\Delta w \geqq q+|dw|^{2}$. This
implies that (3.14), (3.15) and (3.16) are still valid. If we put \( \lambda = 1 \) in (3.15), we obtain \( r = 0 \) and \( u = \text{constant} \) and therefore \( q_\alpha \leq 0 \) for \( \alpha = 1, 2, \cdots, 2n + 2 \). This, together with \( \sum_\alpha q_\alpha = 0 \), implies \( q_\alpha = 0 \) for \( \alpha = 1, 2, \cdots, 2n + 2 \). Therefore, by the same argument as the proof of (\text{*}), we have Theorem 2.

**Remark 5.** After the completion of this paper, the author has been informed that M. Dajczer and G. Thorbergsson ("Holomorphicity of minimal submanifolds in complex space forms", preprint) also obtained our Theorem 1 and Proposition 2, independently.

**References**


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