Geometry of ordinary helices in a complex projective space

(Dedicated to Professor Katsuei Kenmotsu on the occasion of his 60th birthday)

Toshiaki Adachi, Sadahiro Maeda and Seiichi Udagawa
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Abstract. It is well-known that every ordinary helix in Euclidean space is an open curve without self-intersection points. In this paper we study ordinary helices on Riemannian homogeneous spaces. We present an example of closed ordinary helices in a complex projective plane with 6 self-intersection points. We also characterize real space forms in terms of ordinary helices.

Key words: ordinary helix, complex projective plane, self-intersection points.

1. Introduction

Investigating nice curves is important in Differential Geometry as well as in Topology. We pay attention to ordinary helices in a Euclidean 3-space \( \mathbb{R}^3 \). It is well-known that each ordinary helix \( \gamma = \gamma(s) \) parametrized by its arclength \( s \) is an open curve without self-intersection points and satisfies the following ordinary differential equations with respect to \( s \), which are called the Frenet formula for \( \gamma \):

\[
\dot{\gamma} = \kappa V_2, \quad \dot{V}_2 = -\kappa V_1 + \tau V_3, \quad \dot{V}_3 = -\tau V_2.
\]

(1.1)

Here \( \kappa \) and \( \tau \) are positive constants, and \( \{ V_1 = \dot{\gamma}, V_2, V_3 \} \) is an orthonormal frame along \( \gamma \). We call those constants \( \kappa \) and \( \tau \) the curvature and the torsion of \( \gamma \), respectively, and \( \{ V_1, V_2, V_3 \} \) its Frenet frame. The unit vectors \( V_2 \) and \( V_3 \) are called the unit principal normal vector and the unit binormal vector on the curve \( \gamma \), respectively.

On a Riemannian manifold \( (M, \langle , \rangle) \) with Riemannian metric \( \langle , \rangle \), using the covariant differentiation \( \nabla \) induced from the metric \( \langle , \rangle \), we can consider an ordinary helix \( \gamma \) as a solution of a system of ordinary differential
equations corresponding to (1.1). Needless to say, on every Riemannian manifold $M$, for an arbitrary pair $(\kappa, \tau)$ of two positive constants there exists an ordinary helix on $M$ with curvature $\kappa$ and torsion $\tau$. However, in general such ordinary helices are not necessarily congruent each other under the action of the isometry group $G$ of $M$.

In this paper we study ordinary helices in a nonflat $n$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c$, that is, $M_n(c)$ is complex analytically isometric to either a complex projective space $\mathbb{C}P^n(c)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature $c$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ endowed with the Bergman metric of constant holomorphic sectional curvature $c$. After reviewing the definitions and some results on ordinary helices we first construct an example of closed ordinary helices in a complex projective plane $\mathbb{C}P^2(c)$ with 6 self-intersection points (Theorem 1). Later in the class of Riemannian homogeneous spaces we provide a characterization of an $n$-dimensional real space form $M^n(c)$ of constant curvature $c$ in terms of ordinary helices (Theorem 2). Here $M^n(c)$ is isometric to a Euclidean space $\mathbb{R}^n$, a standard sphere $S^n(c)$ with radius $1/\sqrt{c}$ or a real hyperbolic space $H^n(c)$ of constant sectional curvature $c$.

2. Ordinary helices in a Riemannian manifold

Let $M$ be an $n$-dimensional Riemannian manifold with Riemannian connection $\nabla$. We shall start by reviewing the definitions of circles and ordinary helices in $M$. A smooth curve $\gamma = \gamma(s)$ on $M$ parametrized by its arclength $s$ is called a circle, if there exists a field of unit vectors $V_2(s)$ along $\gamma$ which satisfies, together with the field of unit tangent vectors $V_1 = \dot{\gamma}$, the differential equations:

$$\nabla_{\gamma} \dot{\gamma} = \kappa V_2, \quad \nabla_{\gamma} V_2 = -\kappa V_1,$$

where $\kappa$ is a positive constant, which is called the curvature of $\gamma$, and $\nabla_{\gamma}$ denotes the covariant differentiation along $\gamma$ with respect to $\nabla$. We call a smooth curve $\gamma = \gamma(s)$ on $M^n$ with $n \geq 3$ an ordinary helix if there exist an orthonormal frame $\{V_1 = \dot{\gamma}, V_2, V_3\}$ along $\gamma$ and positive constants $\kappa$, $\tau$ which satisfy the following system of ordinary equations
\begin{equation}
\begin{aligned}
\nabla \gamma' &= \kappa V_2, \\
\nabla \gamma V_2 &= -\kappa V_1 + \tau V_3, \\
\nabla \gamma V_3 &= -\tau V_2.
\end{aligned}
\end{equation}

The orthonormal frame \(\{V_1, V_2, V_3\}\) is called the Frenet frame of \(\gamma\), those constants \(\kappa\) and \(\tau\) are called the curvature and the torsion of \(\gamma\), respectively. Note that if \(M\) is a real analytic Riemannian manifold then every ordinary helix is a real analytic curve. The following is a well-known result on the existence and uniqueness of ordinary helices on a Riemannian manifold.

**Proposition A** Given a point \(x \in M\), an orthonormal system \(\{v_1, v_2, v_3\}\) of the tangent space \(T_x M\) and two positive constants \(\kappa, \tau\), we have a unique ordinary helix \(\gamma = \gamma(s)\) on \(M\) having curvature \(\kappa\) and torsion \(\tau\) with \(\gamma(0) = x\) whose initial frame \((V_1(0), V_2(0), V_3(0))\) is \((v_1, v_2, v_3)\).

This guarantees the following result on ordinary helices on a real space form, which is either a standard sphere \(S^n(c)\), a Euclidean space \(\mathbb{R}^n\), or a real hyperbolic space \(H^n(c)\).

**Proposition 1** For an arbitrary pair \((\kappa, \tau)\) of positive constants there exists a unique ordinary helix on a real space form \(M^n(c)\) with curvature \(\kappa\) and torsion \(\tau\) up to the action of the isometry group of \(M^n(c)\).

We shall call two ordinary helices \(\gamma_1, \gamma_2\) on \(M\) are congruent under the action of the isometry group of \(M\) if there exists an isometry \(\varphi\) with \(\gamma_2 = \varphi \circ \gamma_1\). This proposition shows that on a real space form \(M^n(c)\) we have a one-to-one correspondence between the set \(\{(\kappa, \tau) \mid \kappa, \tau > 0\}\) in \(\mathbb{R}^2\) and the set of all congruency classes of ordinary helices under the action of the isometry group.

Here we study ordinary helices on a real space form in connection with the action of the isometry group. On a Euclidean space \(\mathbb{R}^3\) the equation (2.1) is equivalent to the equation \(\gamma'' + (\kappa^2 + \tau^2) \gamma = 0\). If we define an isometry \(\varphi_s\) of \(\mathbb{R}^3\) by \(\varphi_s(x) = A(s) \cdot x + v(s)\), where

\[
A(s) = \begin{pmatrix}
\cos \sqrt{\kappa^2 + \tau^2} s & -\sin \sqrt{\kappa^2 + \tau^2} s & 0 \\
\sin \sqrt{\kappa^2 + \tau^2} s & \cos \sqrt{\kappa^2 + \tau^2} s & 0 \\
0 & 0 & 1
\end{pmatrix} \in \text{SO}(3),
\]
\[ v(s) = \begin{pmatrix} 0 \\ 0 \\ \tau s/\sqrt{\kappa^2 + \tau^2} \end{pmatrix}, \]

every ordinary helix \( \gamma \) is congruent to an ordinary helix \( s \mapsto \varphi_s^{(t}(\kappa/(\kappa^2 + \tau^2), 0, 0)) \).

Next we represent a standard sphere \( S^3(1) \) of curvature 1 as a unit sphere in \( \mathbb{R}^4 \). Regarding an ordinary helix \( \gamma \) on \( S^3(1) \) as a curve on \( \mathbb{R}^4 \), we find the equation (2.1) is equivalent to the equation \( \gamma^{(4)} + (\kappa^2 + \tau^2 + 1) \dot{\gamma} + \tau^2 \ddot{\gamma} = 0 \). Hence if we define an isometry \( \varphi_s \) of \( S^3(1) \) by \( \varphi_s(x) = A(s) \cdot x \) with

\[
A(s) = \begin{pmatrix}
\cos \alpha s & -\sin \alpha s & 0 & 0 \\
\sin \alpha s & \cos \alpha s & 0 & 0 \\
0 & 0 & \cos \beta s & -\sin \beta s \\
0 & 0 & \sin \beta s & \cos \beta s \\
\end{pmatrix} \in \text{SO}(4),
\]

where

\[
\alpha = \frac{1}{2} \left( \sqrt{\kappa^2 + (\tau + 1)^2} + \sqrt{\kappa^2 + (\tau - 1)^2} \right),
\]
\[
\beta = \frac{1}{2} \left( \sqrt{\kappa^2 + (\tau + 1)^2} - \sqrt{\kappa^2 + (\tau - 1)^2} \right),
\]

we see \( \gamma \) is congruent to an ordinary helix \( s \mapsto \varphi_s^{(t}(a, 0, b, 0) \) with

\[
a = \frac{\sqrt{(\beta^2 - 1)^2 + \kappa^2}}{\alpha^2 - \beta^2}, \quad b = \frac{\sqrt{(\alpha^2 - 1)^2 + \kappa^2}}{\alpha^2 - \beta^2}.
\]

Finally we represent a real hyperbolic space \( H^3(-1) \) of curvature \(-1\) as a subset \( \{ x \in \mathbb{R}^4_1 \mid \langle x, x \rangle = -1 \} \) of a de Sitter space \( \mathbb{R}^4_1 \) equipped with the inner product \( \langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 \) for \( x = (x_0, x_1, x_2, x_3) \) and \( y = (y_0, y_1, y_2, y_3) \). Regarding an ordinary helix \( \gamma \) on \( H^3(-1) \) as a curve on \( \mathbb{R}^4_1 \), we find the equation (2.1) is equivalent to the equation \( \gamma^{(4)} + (\kappa^2 + \tau^2 - 1) \dot{\gamma} - \tau^2 \ddot{\gamma} = 0 \). Hence if we define an isometry \( \varphi_s \) of \( H^3(-1) \) by \( \varphi_s(x) = A(s) \cdot x \) with

\[
A(s) = \begin{pmatrix}
\cosh \alpha s & \sinh \alpha s & 0 & 0 \\
\sinh \alpha s & \cosh \alpha s & 0 & 0 \\
0 & 0 & \cos \beta s & -\sin \beta s \\
0 & 0 & \sin \beta s & \cos \beta s \\
\end{pmatrix} \in \text{O}(1, 3),
\]
where
\[
\alpha = \frac{\sqrt{2}}{2} \sqrt{\sqrt{(\kappa^2 + \tau^2 + 1)^2 + 4\tau^2 + 1} - \kappa^2 - \tau^2},
\]
\[
\beta = \frac{\sqrt{2}}{2} \sqrt{\kappa^2 + \tau^2 - 1 + \sqrt{(\kappa^2 + \tau^2 + 1)^2 + 4\tau^2},}
\]
we see \(\gamma\) is congruent to an ordinary helix \(s \mapsto \varphi_s(t(a,0,b,0))\) with
\[
a = \frac{\sqrt{(\beta^2 + 1)^2 - \kappa^2}}{\alpha^2 + \beta^2}, \quad b = \frac{\sqrt{\kappa^2 - (\alpha^2 - 1)^2}}{\alpha^2 + \beta^2}.
\]

**Proposition 2** Every ordinary helix \(\gamma = \gamma(s)\) on a real space form \(M^n(c)\) is an integral curve of some Killing vector field on \(M^n(c)\), that is, there exists a one-parameter subgroup \(\{\varphi_s\}\) of the isometry group of \(M^n(c)\) with \(\gamma(s) = \varphi_s(\gamma(0))\) for all \(s\).

We now concern ourselves on ordinary helices on a complex space form. We remark that corresponding assertions in Propositions 1 and 2 do not hold for ordinary helices on a nonflat complex space form \(M_n(c)\) with complex structure \(J\). In order to explain this fact in detail we use the complex torsions \(t_{ij}(s) = \langle V_i(s), JV_j(s) \rangle\) of an ordinary helix \(\gamma\), where \(1 \leq i < j \leq 3\) and \(\{V_1, V_2, V_3\}\) is the Frenet frame for \(\gamma\). Note that every complex torsion \(t_{ij} = t_{ij}(s)\) of each ordinary helix is a real analytic function and satisfies \(|t_{ij}(s)| \leq 1\) for every \(s\). In this paper we shall call a smooth curve \(\gamma\) on \(M_n(c)\), \(c \neq 0\) a Killing ordinary helix if it is an ordinary helix and also is an integral curve of some Killing vector field on \(M_n(c)\). For the later use we recall the following.

**Theorem A** ([MA]) Let \(M_n(c)\) be an \(n (\geq 3)\)-dimensional nonflat complex space form. Then the following hold:

1. All complex torsions of each Killing ordinary helix \(\gamma\) in \(M_n(c)\) are constant. They satisfy
\[
\kappa t_{23} = \tau t_{12}, \quad t_{13} = 0, \quad |t_{12}| \leq \kappa/\sqrt{\kappa^2 + \tau^2},
\]
with curvature \(\kappa\) and complex torsion \(\tau\) of \(\gamma\).

2. Conversely, given positive constants \(\kappa, \tau\) and a constant \(\mu\) with \(|\mu| \leq \kappa/\sqrt{\kappa^2 + \tau^2}\), we have a unique Killing ordinary helix on \(M_n(c)\) up to the action of holomorphic isometries of \(M_n(c)\) whose curvature is \(\kappa\), torsion is \(\tau\), and the first complex torsion \(t_{12}\) is \(\mu\).
(3) If $|\mu| > \kappa/\sqrt{\kappa^2 + \tau^2}$, we have no such a Killing ordinary helix on $M_n(c)$.

**Theorem B** ([MA]) Let $M_2(c)$ be a 2-dimensional nonflat complex space form. Then the following hold:

1. All complex torsions of each Killing ordinary helix $\gamma$ in $M_2(c)$ are constant. They are expressed as either
   \[\tau_{12} = \kappa/\sqrt{\kappa^2 + \tau^2}, \quad \tau_{13} = 0, \quad \tau_{23} = \tau/\sqrt{\kappa^2 + \tau^2}\]  
   (2.2)
   or
   \[\tau_{12} = -\kappa/\sqrt{\kappa^2 + \tau^2}, \quad \tau_{13} = 0, \quad \tau_{23} = -\tau/\sqrt{\kappa^2 + \tau^2}\]  
   (2.3)
   by use of curvature $\kappa$ and torsion $\tau$ of $\gamma$.

2. Conversely, for given positive constants $\kappa$, $\tau$ we have two Killing ordinary helices on $M_2(c)$ whose curvature is $\kappa$ and torsion is $\tau$ up to the action of holomorphic isometries on $M_2(c)$. Their complex torsions correspond to either (2.2) or (2.3).

It is well-known that every isometry of a nonflat complex space form $M_n(c)$ is either holomorphic or anti-holomorphic with respect to the given complex structure $J$ of $M_n(c)$. We should note that those Killing ordinary helices having the same curvature $\kappa$ and torsion $\tau$ with complex torsions (2.2) or (2.3) are mutually congruent by some anti-holomorphic isometry of $M_n(c)$.

Similarly those statements in Proposition 2 do not hold for ordinary helices on other symmetric spaces of rank one, which are a quaternionic projective space $\mathbb{H}P^n$, a quaternionic hyperbolic space $\mathbb{HH}^n$, a Cayley plane $\mathbb{O}P^2$ and a Cayley hyperbolic plane $\mathbb{OH}^2$. We denote by $\mathbb{K}$ either the field of quaternion numbers $\mathbb{H}$ or the Cayley algebra $\mathbb{O}$. For an ordinary helix $\gamma$ on $\mathbb{K}P^n$ or $\mathbb{K}H^n$ we put $\tau_{ij}(s) = \|\text{Proj}_j(V_i(s))\|$, $1 \leq i < j \leq 3$, where $\text{Proj}_j$ denotes the projection of the tangent space at $\gamma(s)$ onto the $\mathbb{K}$-subspace generated by $V_j(s)$. These correspond to the absolute value of the complex torsions for ordinary helices on nonflat complex space forms. On these spaces $\mathbb{K}P^n$ and $\mathbb{K}H^n$ every isometry $\varphi$ preserves the $\mathbb{K}$-structure; that is, $d\varphi$ maps the $\mathbb{K}$-subspace spanned by a tangent vector $v$ to the $\mathbb{K}$-subspace spanned by $d\varphi(v)$. Therefore if an ordinary helix $\gamma$ on $\mathbb{K}P^n$ or $\mathbb{K}H^n$ is generated by some Killing vector field on such a space, then $\tau_{ij}$ should be constant along $\gamma$. We now take for example an ordinary helix of curvature $\kappa$
and of torsion $\tau$ on $\mathbb{C}P^2$ or $\mathbb{CH}^2$ with $\tau_{13}(0) \neq 0$. Then $\tau_{12}$ is not constant, because $\tau_{12}(s) = \tau \cdot \tau_{13}(s)$. Hence it follows from the fact that $\mathbb{K}P^n$ admits $\mathbb{C}P^2$ as a totally geodesic submanifold and $\mathbb{K}H^n$ admits $\mathbb{C}H^2$ as a totally geodesic submanifold and Theorem A that the following result holds.

**Proposition 3** Let $M$ be a Riemannian symmetric space of rank one which is not a real space form, and $(\kappa, \tau)$ be a pair of positive numbers.

1. There exist on $M$ infinitely many congruency classes of ordinary helices which are not Killing ordinary helices whose curvature is $\kappa$ and whose torsion is $\tau$.

2. When $M$ is not a plane, that is, $M \neq \mathbb{C}P^2, \mathbb{HP}^2, \mathbb{OP}^2, \mathbb{CH}^2, \mathbb{HH}^2, \mathbb{OH}^2$, there exist on $M$ infinitely many congruency classes of Killing ordinary helices whose curvature is $\kappa$ and whose torsion is $\tau$.

3. Construction of closed ordinary helices with self-intersections in $\mathbb{C}P^2$

In this section we construct a closed ordinary helix with self-intersection points on a complex projective plane $\mathbb{C}P^2$ by using Riemannian submanifold theory. We consider a quotient of a 2-dimensional flat torus $N = (S^1 \times S^1)/\phi$. Here representing the first component by $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the second component by $S^2 = \{(a_1, a_2) \in \mathbb{R}^2 \mid (a_1)^2 + (a_2)^2 = 1\}$, we define the identification $\phi$ by $\phi((e^{i\theta}, (a_1, a_2))) = (-e^{i\theta}, (-a_1, -a_2))$. The Riemannian metric on $N$ is induced by the metric on a torus $S^1 \times S^1$ which is given by $\langle A + \xi, B + \eta \rangle = \frac{2}{9} \langle A, B \rangle_{S^1} + \frac{2}{3} \langle \xi, \eta \rangle_{S^1}$ for tangent vectors $A, B \in TS^1$ of the first component and tangent vectors $\xi, \eta \in TS^1$ of the second component, where $\langle \cdot, \cdot \rangle_{S^1}$ denotes the canonical metric on $S^1$.

In order to obtain an ordinary helix with self-intersection points we make use of the following isometric embedding $f : N \to \mathbb{C}P^2(4)$ defined by

$$f([e^{i\theta}, (a_1, a_2)]) = \pi \left( \frac{1}{3}(e^{-\frac{2i\theta}{3}} + 2a_1 e^{\frac{2i\theta}{3}}), \frac{\sqrt{2}}{3}(e^{-\frac{2i\theta}{3}} - a_1 e^{\frac{2i\theta}{3}}), \frac{2}{\sqrt{6}}ia_2 e^{\frac{i\theta}{3}} \right), \quad (3.1)$$

where $\pi : S^5(1) \to \mathbb{C}P^2(4)$ is the Hopf fibration. By virtue of the result in [AM], the image $f \circ \gamma$ of a circle $\gamma$ of curvature $1/2$ on $N$ is an ordinary helix on $\mathbb{C}P^2(4)$ with curvature $\sqrt{3}/2$ and torsion $\sqrt{3}/2$. The complex torsions of $f \circ \gamma$ are described as
\[ \tau_{12} = \sqrt{\frac{2}{3}} \cos \left( \frac{1}{2}s + \psi_0 \right), \quad \tau_{13} = -\sin \left( \frac{1}{2}s + \psi_0 \right), \]
\[ \tau_{23} = -\frac{1}{\sqrt{3}} \cos \left( \frac{1}{2}s + \psi_0 \right). \]

Here, \( \psi_0 \) is the angle between \( \gamma'(0) \) and the unit vector \( u \) tangent to the first component of \( N \). As the complex torsions of \( f \circ \gamma \) are not constants, we find it is not an integral curve of any Killing vector field on \( \mathbb{CP}^2(4) \).

We now consider the universal Riemannian covering \( p : \mathbb{R}^2 \to N \). Regarding the Riemannian metric on \( N \), we can choose a fundamental region for \( N \) in \( \mathbb{R}^2 \) as \( \mathcal{F} = [0, 2\sqrt{2}\pi/3) \times [0, \sqrt{6}\pi/3) \). Two points \((x_1, x_2)\) and \((y_1, y_2)\) on \( \mathbb{R}^2 \) satisfy \( p((x_1, x_2)) = p((y_1, y_2)) \) if and only if they satisfy one of the following conditions:

i) \( x_1 - y_1 = 2\sqrt{2}m_1\pi/3, \quad x_2 - y_2 = 2\sqrt{6}m_2\pi/3 \) for some \( m_1, m_2 \in \mathbb{Z} \),

ii) \( x_1 - y_1 = \sqrt{2}(2m_1 + 1)\pi/3, \quad x_2 - y_2 = \sqrt{6}(2m_2 + 1)\pi/3 \) for some \( m_1, m_2 \in \mathbb{Z} \).

Let \( \tilde{\gamma} \) denote a covering circle in \( \mathbb{R}^2 \) of a circle \( \gamma \) of curvature 1/2 on \( N \). Then the curve \( \tilde{\gamma} \) is a circle of radius 2 in the sense of Euclidean Geometry. This, together with the fact that \( f \) is an isometric imbedding, shows that the curve \( f \circ \gamma \) is a closed curve of length 4\( \pi \) in \( \mathbb{CP}^2(4) \). Moreover, since the curvature 1/2 of the circle \( \gamma \) is less than \( 3/(\sqrt{2}\pi) \), it has self-intersection points and hence so does the ordinary helix \( f \circ \gamma \). Suppose \( \gamma(s_0) \) is a self-intersection point. Denoting the tangential vector \( \gamma'(s_0) \) by \((v_1, v_2) \in \mathbb{R}^2 \cong T_{\gamma(s_0)}N \), we have

\[ \tilde{\gamma}'(s) = 2 \begin{pmatrix} v_1 \sin \frac{s - s_0}{2} + v_2 \left( \cos \frac{s - s_0}{2} - 1 \right) \\ v_2 \sin \frac{s - s_0}{2} - v_1 \left( \cos \frac{s - s_0}{2} - 1 \right) \end{pmatrix} + \tilde{\gamma}(s_0). \]

If \( \gamma(s_0 + s_1) = \gamma(s_0) \), then we get in the case i) that

\[
\begin{align*}
  v_1 \sin \frac{s_1}{2} + v_2 \left( \cos \frac{s_1}{2} - 1 \right) &= \frac{\sqrt{2}}{3} m_1 \pi, \\
  v_2 \sin \frac{s_1}{2} - v_1 \left( \cos \frac{s_1}{2} - 1 \right) &= \frac{\sqrt{6}}{3} m_2 \pi.
\end{align*}
\]

As \( v_1^2 + v_2^2 = 1 \), we obtain in this case that
\[
\begin{align*}
\sin \frac{s_1}{2} &= \frac{\sqrt{2}}{3} \pi (m_1 v_1 + \sqrt{3}m_2 v_2), \\
\cos \frac{s_1}{2} &= \frac{\sqrt{2}}{3} \pi (m_1 v_2 - \sqrt{3}m_2 v_1) + 1.
\end{align*}
\]

Thus we find \((v_1, v_2)\) satisfies either

I) \(\pi(m_1^2 + 3m_2^2) = 3(-\sqrt{2}v_2m_1 + \sqrt{6}v_1m_2)\), or

II) \(\pi\{(2m_1 + 1)^2 + 3(2m_2 + 1)^2\} = 3(-\sqrt{2}v_2(2m_1 + 1) + \sqrt{6}v_1(2m_2 + 1)\}

for some pair of integers \((m_1, m_2) \neq (0, 0)\) corresponding to the conditions i) and ii). In our case, as the curvature 1/2 of \(\gamma\) is greater than \(3/(\sqrt{6}\pi)\), such conditions might occur only for \((m_1, m_2) = (\pm 1, 0)\) for the case I) and for \((m_1, m_2) = (0, -1), (0, 0), (-1, 0), (-1, -1), (1, -1), (1, 0)\) for the case II) (see Figure 1). Checking these conditions (or looking Figure 1 carefully), we find \(\gamma\) does not have self-intersection points corresponding to \((m_1, m_2) = (1, -1), (1, 0)\) for II), and has self-intersection points corresponding to other conditions. Thus we find \(\gamma\) has 6 self-intersection points, and so does \(f \circ \gamma\).

\[\text{Fig. 1. Circle of curvature 1/2 on } N\]
\((P(2\sqrt{2}\pi/3, 0), Q(0, \sqrt{6}\pi/3), A(2, 0), B(0, 2))\).
Summarizing up we obtain the following.

**Theorem 1** Let $f : N \rightarrow \mathbb{C}P^2(4)$ denote the isometric embedding defined by (3.1) and $\gamma$ be a circle of curvature $1/2$ on $N$. Then the curve $f \circ \gamma$ satisfies the following:

1. It is a closed ordinary helix of length $4\pi$ with curvature $\sqrt{3}/2$ and torsion $\sqrt{3}/2$ on $\mathbb{C}P^2(4)$.
2. It is not generated by any Killing vector field on $\mathbb{C}P^2(4)$ and has 6 self-intersection points.

**Remark 1** In connection with submanifolds we have the following Killing ordinary helix on $\mathbb{C}P^2(4)$. Let $h : \mathbb{C}P^1(2) (= S^2(2)) \rightarrow \mathbb{C}P^2(4)$ denote the second Veronese embedding given by $h([z_0, z_1]) = [z_0^2, \sqrt{2}z_0z_1, z_1^2]$ with homogeneous coordinate $(z_0, z_1)$ on $\mathbb{C}P^1(2)$. For a circle of curvature $1/\sqrt{2}$ on $\mathbb{C}P^1(2)$, the curve $h \circ \gamma$ is a Killing ordinary helix of curvature $\sqrt{3}/2$ and torsion $\sqrt{3}$. This ordinary helix is closed with length $2\sqrt{2/5\pi}$ and does not have self-intersection points (see [AM]).

4. **Characterization of real space forms by ordinary helices**

A Riemannian manifold $M$ is said to be homogeneous if the isometry group $G$ of $M$ acts transitively on $M$. Our study in this section is motivated by the following theorem:

**Theorem C** ([MT]) Let $M$ be a Riemannian homogeneous space and $\kappa$ be a positive constant. If every circle of curvature $\kappa$ on $M$ is generated by some Killing vector field on $M$, then $M$ is either a Euclidean space or a Riemannian globally symmetric space of rank one.

Our aim here is to provide the following characterization of real space forms by ordinary helices in the class of Riemannian homogeneous spaces.

**Theorem 2** Let $M$ be a Riemannian homogeneous space and $\kappa$, $\tau$ be positive constants. Every ordinary helix of curvature $\kappa$ and torsion $\tau$ on $M$ is generated by some Killing vector field on $M$ if and only if $M$ is a real space form.

Let $K$ denote the compact isotropy subgroup of $G$ such as $M = G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. Take an Ad($K$)-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. For each $X \in \mathfrak{p}$, we define an element $\Lambda_p(X) \in \mathfrak{so}(\mathfrak{p})$ by
\[\Lambda_p(X)(Y) = \frac{1}{2} [X, Y]_p + U(X, Y)\]

for \(Y \in p\), where \(U : p \times p \to p\) is given by the equality

\[\langle U(X, Y), Z \rangle = \frac{1}{2} \{\langle[Z, X]_p, Y \rangle + \langle[Z, Y]_p, X \rangle\}\]

for every \(X, Y, Z \in p\). We define a linear map \(\Lambda : g \to \mathfrak{so}(p)\) by \(\Lambda(X) = \text{ad}_{X_t} + \Lambda_p(X_p)\), where for \(X \in g\) we denote by \(X = X_t + X_p\) with \(X_t \in \mathfrak{k}\) and \(X_p \in p\). We denote the projection by \(\varpi : G \to G/K\). Along the same lines as in [MT] we can conclude the following.

**Proposition 4** Let \(\{X, Y, Z\}\) be a triplet of orthonormal vectors in \(p\) and \(H\) be an element of \(\mathfrak{k}\). The orbit \(\gamma(t) = \varpi(\exp t(H + X))\) is an ordinary helix on \(G/K\) of curvature \(\kappa\) and torsion \(\tau\) with initial frame \((X, Y, Z)\) if and only if the following equalities hold:

\[\Lambda(H + X)(X) = [H, X] + \Lambda_p(X)(X) = \kappa Y,\]
\[\Lambda(H + X)(Y) = [H, Y] + \Lambda_p(X)(Y) = -\kappa X + \tau Z,\]
\[\Lambda(H + X)(Z) = [H, Z] + \Lambda_p(X)(Z) = -\tau Y.\]

**Corollary** Let \(M = G/K\) be a Riemannian symmetric space. Let \(\{X, Y, Z\}\) be a triplet of orthonormal vectors in \(p\) and \(H\) be an element of \(\mathfrak{k}\). The orbit \(\gamma(t) = \varpi(\exp t(H + X))\) is an ordinary helix on \(G/K\) of curvature \(\kappa\) and torsion \(\tau\) with initial frame \((X, Y, Z)\) if and only if the following equalities hold:

\[[H, X] = \kappa Y, \quad [H, Y] = -\kappa X + \tau Z, \quad [H, Z] = -\tau Y.\]

**Proof of Theorem 2.** The “if” part of the theorem is Proposition 2. In the following we prove the “only if” part. Suppose that every ordinary helix with curvature \(\kappa\) and torsion \(\tau\) in \(M = G/K\) is an orbit of one-parameter subgroup of \(G\). Let \(\{X, Y, Z\}\) be an arbitrary triplet of mutually orthonormal vectors in \(p\). Take two ordinary helices \(\gamma_1\) and \(\gamma_2\) of curvature \(\kappa\) and torsion \(\tau\) which go through the point \(eK = \gamma_1(0) = \gamma_2(0)\) and whose initial frames \((V_1(0), V_2(0), V_3(0))\) are \((X, Y, Z)\) and \((X, -Y, Z)\), respectively. By Proposition 4 we have \(H_1, H_2 \in \mathfrak{k}\) with

\[[H_1, X] + \Lambda_p(X)(X) = \kappa Y, \quad [H_2, X] + \Lambda_p(X)(X) = -\kappa Y.\]

We then obtain \([H, X] = Y\) for \(H = (1/2\kappa)(H_1 - H_2)\). Repeating the same discussion as in the proof of Theorem 3.1 in [MT], we find that the linear
isotropy group $\text{Ad}(K)$ acts transitively on the unit sphere $S(p)$ in $p$. Hence $M$ is an Euclidean space or a Riemannian globally symmetric space of rank one.

On each Riemannian symmetric space of rank one which is not a real space form we have by Proposition 3 ordinary helices of curvature $\kappa$ and torsion $\tau$ which are not generated by any Killing vector field on this space. We hence get our conclusion. 

Before closing this paper, we write down the form of an ordinary helix on a symmetric space $G/K$. For a curve $\gamma : \mathbb{R} \to G/K$ we represent its framing $F_\gamma : \mathbb{R} \to G$, which is a map with $\gamma = \varphi \circ F_\gamma$, as $F_\gamma = g_\gamma h_\gamma$ by using maps $g_\gamma : \mathbb{R} \to G$, $h_\gamma : \mathbb{R} \to K$ with the property that $g_\gamma^{-1}(dg_\gamma/dt) \in \mathfrak{m}$.

**Proposition 5** A smooth curve $\gamma$ on a Riemannian symmetric space $G/K$ is an ordinary helix of curvature $\kappa$ and torsion $\tau$ with initial frame $(X, Y, Z)$ if and only if $g_\gamma$ is a solution of the following differential equation with $g_\gamma(0) = I$:

$$
g_\gamma^{-1} \frac{dg_\gamma}{dt} = \frac{1}{\kappa^2 + \tau^2}(\kappa^2 \cos \sqrt{\kappa^2 + \tau^2}^2 + \tau^2)X + \frac{\kappa \sin \sqrt{\kappa^2 + \tau^2}}{\sqrt{\kappa^2 + \tau^2}}Y + \frac{\kappa \tau}{\kappa^2 + \tau^2}(1 - \cos \sqrt{\kappa^2 + \tau^2}^2)Z. \quad (4.1)
$$

In this case, each entry of the solution $g_\gamma(t)$ is represented by the power series of $t$ with infinite radius of convergence.

**Proof.** Since we can easily get the last claim by standard argument (cf. [AMU]), we only show the "only if" part. Suppose $\gamma$ is an ordinary helix. Let $F_\gamma^{-1}(dF_\gamma/dt) = \alpha_\xi + \alpha_\mathfrak{m}$ be the decomposition according as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. By using $g_\gamma$ and $h_\gamma$ we have

$$
F_\gamma^{-1}(dF_\gamma/dt) = h_\gamma^{-1}(dh_\gamma/dt) + \text{Ad} h_\gamma^{-1}(g_\gamma^{-1}(dg_\gamma/dt)).
$$

We here note that $\alpha_\xi = h_\gamma^{-1}(dh_\gamma/dt)$ always has a solution for a given $\alpha_\xi$ (see [KN]). We then have $\alpha_\mathfrak{m} = \text{Ad} h_\gamma^{-1}(g_\gamma^{-1}(dg_\gamma/dt))$ for such a solution $h_\gamma$.

By the equation (2.1) we find the curve $\tilde{\alpha}_\mathfrak{m}$ which is defined by $\text{Ad} h_\gamma \alpha_\mathfrak{m}$ satisfies

$$
\frac{d^3 \tilde{\alpha}_\mathfrak{m}}{dt^3} = -(\kappa^2 + \tau^2) \frac{d\tilde{\alpha}_\mathfrak{m}}{dt},
$$
hence we see by initial condition $\nabla_\gamma \nabla_\gamma \gamma(0) = -\kappa^2 X + \kappa \tau Z$ that
\[
\frac{d^2 \tilde{\alpha}_m}{dt^2} = - (\kappa^2 + \tau^2) \tilde{\alpha}_m + \tau^2 X + \kappa \tau Z.
\]
If we set
\[
\tilde{\alpha}_m = \tilde{\alpha}_m - \frac{1}{\kappa^2 + \tau^2} (\tau^2 X + \kappa \tau Z),
\]
then we have $d^2 \tilde{\alpha}_m / dt^2 = - (\kappa^2 + \tau^2) \tilde{\alpha}_m$. Solving this equation, we obtain
\[
\tilde{\alpha}_m = \cos \sqrt{\kappa^2 + \tau^2} t \left( X - \frac{1}{\kappa^2 + \tau^2} (\tau^2 X + \kappa \tau Z) \right) + \frac{\kappa \sin \sqrt{\kappa^2 + \tau^2} t}{\sqrt{\kappa^2 + \tau^2}} Y + \frac{1}{\kappa^2 + \tau^2} (\tau^2 X + \kappa \tau Z),
\]
which lead us to the desirable differential equation because $\tilde{\alpha}_m = g^{-1}_\gamma (d g_{\gamma} / dt)$.

This proposition also shows Corollary. For an orbit $\gamma(t) = \varpi \cdot \exp(t(H + X))$ on a symmetric space $G/K$ we may have $g_\gamma(t) = \exp(t(H + X)) \cdot \exp(-tH)$, hence obtain $g^{-1}_\gamma (d g_{\gamma} / dt) = \text{Ad} \exp(tH) X$. This shows that the curve $\tilde{\alpha}_m = g^{-1}_\gamma (d g_{\gamma} / dt)$ satisfies $(d^2 \tilde{\alpha}_m / dt^2)(0) = (\text{ad} H)^3 X$. On the other hand, by (4.1) we have
\[
\frac{d \tilde{\alpha}_m}{dt}(0) = \kappa Y, \quad \frac{d^2 \tilde{\alpha}_m}{dt^2}(0) = -\kappa^2 X + \kappa \tau Z, \quad \frac{d^3 \tilde{\alpha}_m}{dt^3}(0) = -\kappa (\kappa^2 + \tau^2) Y.
\]
Thus we get Corollary.

References


Toshiaki Adachi  
Department of Mathematics  
Nagoya Institute of Technology  
Nagoya 466-8555, Japan  
E-mail: adachi@nitech.ac.jp

Sadahiro Maeda  
Department of Mathematics  
Shimane University  
Matsue, Shimane 690-8504, Japan  
E-mail: smaeda@math.shimane-u.ac.jp

Seiichi Udagawa  
Department of Mathematics  
School of Medicine  
Nihon University  
Itabashi, Tokyo 173-0032, Japan  
E-mail: sudagawa@med.nihon-u.ac.jp